INTRODUCTION

TO THE

ELEMENTS OF EUCLID

PART I.

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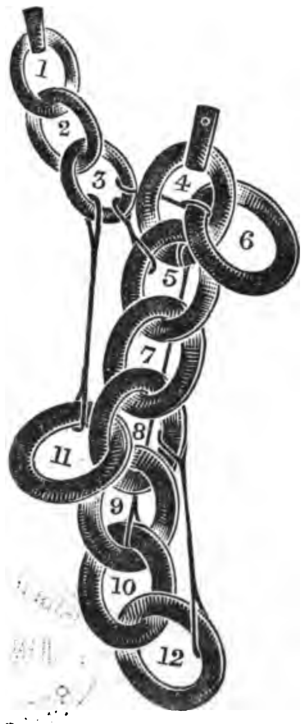
Frontispiece



SHOWING THE INTERDEPENDENCE OF THE FIRST TWELVE PROPOSITIONS OF THE FIRST BOOK OF EUCLID

[*For explanation see page* 124.

AN INTRODUCTION

TO THE

ELEMENTS OF EUCLID

PART I.

BEING A FAMILIAR EXPLANATION

OF THE

FIRST TWELVE PROPOSITIONS OF THE FIRST BOOK

BY THE

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THIRD EDITION

LONDON
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1880

ΜΥΣΤΑΙΣ

SUIS

ἀγαπητοῖς

HOC OPUSCULUM DICAT

S. H.

Extract from 'Narrative Essay on a Liberal Education,' by the writer of this Treatise.—'And not only is the Euclid lesson an interest full of charm,—it forms a bond of firm friendship in after-life. How often, amidst the mountains and valleys of Switzerland, do I find my hand grasped, and, looking up, I see an animated face radiant with pleasant memories. "What, don't you know me? I am one of your old *μύσται*," alluding to a long and growing list, that used to hang up in the Mathematical School at Eton, of the *Initiated*—that is, of those who understood and could apply the Fourth Proposition of the First Book. These were my *μύσται*. The Platonic motto "*μηδεὶς ἀγεωμέτρητος εἰσίτω*" standing at the head of the list shut out the *profanum vulgus*; and hearty—nay, vehement sometimes—were the struggles to get into it.'

PREFACE

TO

THE SECOND EDITION.

In preparing a Second Edition of this treatise, the writer has taken great pains to make it a practically useful school-book.

In comparing the present edition with the former, it will be found that a good deal of it has been re-written; especially the discussion of the seventh and eighth propositions, which some, perhaps, will think too elaborate.

The writer would say in reply to such an objection, that a long experience has taught him that beginners find none, among the early propositions of Euclid, so baffling as the seventh and eighth. After a very long time spent on them, learners, for the most part, show that they have not taken in Euclid's meaning and aim; and so have failed to reap the intellectual benefit which is to be derived from a right appreciation of them.

At the same time he considers it to be very undesirable that learners should lose the intellectual exercise and benefit which a struggle with Euclid's method of proving the eighth proposition through the seventh affords.

Now, by approaching these propositions as is done in this treatise, it is found that one or two lessons, interspersed with pleasant laughter, are quite enough to make learners thoroughly see Euclid's intention; and get a grasp of the propositions, which they never let go.

For more than forty years the writer has found, in teaching Euclid, one of the chief enjoyments of his life, and before his time is ended he very much wishes to leave behind him as good an account as he can of his method, in order that others, coming after him, may derive the same enjoyment that he has done from the work; at the same time that they will be doing very great good to future generations of the boys and girls of England.

He hopes, then, that, under these circumstances, he will not incur the charge of self-sufficiency, if he specifies two or three results of this mode of teaching which tutors will not fail to appreciate.

First it makes the Euclid lesson to be thoroughly enjoyed; instead of its being repulsive, it becomes interesting—often, indeed, a delight to pupils as well as teachers.

In the year 1868 a sketch of the mode of teaching Euclid, embodied in this treatise, was put forth

in an essay describing an attempt to give a liberal education to children of the working class. In reference to that sketch the following words occur in the *Athenæum* of Oct. 24, 1868:—'This mode of teaching Euclid is positively fascinating, as well as thoroughly sound and accurate. It is no wonder that such teaching should make boys like Euclid better than any other lesson.'

Secondly, it tends to remove from our Euclid teaching a defect which is almost universally condemned by examiners. They say that whereas in many cases boys will write out the words of a proposition fairly well—in hardly any case will they venture to look at a rider or deduction.

In this treatise every proposition is approached as an intelligent boy should approach a deduction; so that boys thus taught are at home with exercises and riders; they will fasten on a deduction with interest, look keenly into it, and enjoy wrestling with it.

A third result follows as a matter of course from the foregoing. A valuable foundation is thus laid for the future profitable reading, both of subjects purely mathematical, and of those branches of scientific knowledge which are taught in many schools, and enter largely into the competitive examinations of the present day.

A series of questions has been added as a help to teachers in finding out if their pupils have read the text with thought and intelligence.

A selection of exercises has also been added, not requiring more knowledge of Euclid than may be gained from the present treatise.

S. H.

September 1878.

PREFACE

TO

THE FIRST EDITION.

THE FIRST IMPRESSION of any one who takes up this book may possibly be, how much there is here, and, after all, what a little way it goes in EUCLID—only to the Twelfth Proposition.

But teachers know that the first Twelve Propositions of the First Book is the battle-field with Euclid. If the victory is won here, all is easy afterwards. A keen glance and a little sustained thought is all that is needed to master any subsequent Proposition.

And this being so, the learner's progress through Euclid will be as interesting and profitable as it is useless and unprofitable if he has not learned to appreciate geometrical reasoning, and to enjoy its rigour and satisfactoriness.

Perhaps it will form the best preface to what follows if the writer states, not the object of the treatise only, but the circumstances under which it was written.

A happy connection of thirty-six years with Eton has closed, and as the writer spends the first weeks

of his retirement among the mountains of Switzerland, many heart-moving thoughts about bygone days arise; especially there comes before him the remembrance of very pleasant hours spent with successive generations of loyal-hearted boys over the early Propositions of Euclid.

He is aware that it is the experience of some teachers that most boys find the study of Euclid repulsive: his experience is the reverse of this; he cannot call to mind the case of a single pupil who found the study of Euclid *repulsive.* They may have found it hard. They may not, in some cases, have been industrious and persevering enough to succeed in examinations. But all of them have felt that there was something real and great in Euclid. If they have failed, they have readily acknowledged that their failure was their own fault, and not because there was no meaning in Euclid.

It would be very heartless, too, if the writer were to forget the many visits full of friendliness, and even affection, which he received from pupils who had passed out of his hands—both during their Eton career, and after they had left Eton—to talk over the old days when they began Euclid together, and to tell him the good which the lessons on the first Twelve Propositions had done them.

Now the writer, in teaching Euclid, always made it his special aim, at the first start, to give life, animation, and suppleness to the cold and rigid form in which Euclid is presented to beginners in their school books. This he found to be most successfully

done by familiar talk, appeals to common sense, homely explanations, reference to things of every-day life, and even humorous illustrations, and stories in point.

In the following treatise the substance of such conversations is embodied, with the view of making Euclid a book of life and meaning to those learners who, beginning the study of it in the common, regular editions of the work—in which the style is formal, rigid, accurate—find it *but a book of words.*

What Euclid examiner has not had to mourn over pages of utter trash, mere verbiage, shown up by boy after boy—a feeble imitation of Euclid's phraseology, but giving no sign whatever of the writers having any idea of the meaning of the words which they were using!

Now the aim of this treatise is simply to shake learners out of the miserable habit of resting in the words of Euclid; and to lead them to be dissatisfied till they feel that they are taking in his meaning.

Those beginners who do not need the book need not use it. Let them take in hand any of the good modern editions of Euclid, and grapple with them unaided.

If others find the explanation too profuse and endless; if they find themselves growing weary of the reiteration, let them deal with the book as they do with their corks or swimming-belt. When they want them no longer, they throw them away, and swim alone.

It is the writer's earnest desire to guide and lead

on his readers so patiently and helpfully that not one shall be left behind. It is the slowest and weakest that he especially wishes to take by the hand. The very difficulty they have in taking in Euclid's meaning shows that they most need Euclid.

Such is the writer's object, and his first thought, in putting into print the substance of his conversational method of teaching Euclid, has had reference to his old Eton pupils. He wishes, though no longer able to speak to them face to face, thus to offer to them the helping-hand of which they used to take a friendly grip.

But he looks to others also with whom he has had no personal connection. The many hearty and very friendly letters which he received from teachers both in England and America when, some years ago, he gave in a tract[1] a sketch of this mode of teaching Euclid, leads him to hope that they may find this treatise of use in their schools.

He trusts, too, that those who are struggling with Euclid, without any one at hand to explain it, and to remove difficulties, may find the treatise a help to them. And he will be glad if some, who look askance, with a half-smile and half-shudder at the idea of reading Euclid, shall be led on by these familiar explanations to a study that will be of incalculable benefit to them.

Another set of readers whom the writer has in his mind are those classes of women who, having tried this or that fragment of a scientific subject, have

[1] *A Narrative Essay on a Liberal Education*, Hamilton & Adams.

come to the sound conclusion that they must 'begin at the beginning, and lay a foundation,' and who have, most wisely, chosen Euclid for that foundation.

To these classes he hopes that his treatise may be useful. But as they will be in earnest, and assiduous, a half-apology is due to them for the homely and humorous illustrations introduced to arrest attention and awaken interest. Let them kindly remember that the following explanations were given to, and are written for, boys, who, high-spirited and thoughtless, are not fully alive to the importance of what they are learning, or bent, as they are, on self-improvement.

It has been already said that these explanations are not written for quick and able boys, but for the great mass of those of average intellect, who have not naturally the accuracy of thought, and the powers of reasoning with which some few are gifted. Yet even an able boy may start at an advantage by a rapid perusal of these familiar explanations. Possibly he may gain a fuller view of Euclid's reasoning than he would have had without them, and pick up thoughts that he might otherwise have missed.

One word more: a body of men, able mathematicians, are now working together with a view of improving our methods of geometrical teaching. The writer deprecates the charge of antagonism with them.

They are to a great degree entering on that profession which he is leaving. Euclid was king in the writer's time, was still seated on the throne which

he has occupied for more than two thousand years. If the common consent of the leading mathematicians of England, in the next generation, decide that he must be dethroned, he must go. But geometrical reasoning will not go. And as the aim of this treatise is to make learners appreciate geometrical reasoning, the writer counts on the sympathy of all who are desirous of improving our geometrical teaching and are working towards that end.

A WORD of ADVICE to the LEARNER.

PROVIDE YOURSELF with a pair of compasses having a pen and pencil leg, a small flat ruler, and a hard pencil (and keep its point fine). Also it would be well to have by you a little box-wood triangle, called by instrument makers a 'set square.' In default of this, double down and cut off the corner of a sheet of cardboard; it will answer the same purpose.

While you are reading this treatise, have your pen much in your hand, and often try if you understand, and remember what you have read, by writing it out, not copying it.

Remember Bacon's apophthegm:—

> Much reading makes a full man,
> Much speaking makes a ready man,
> Much writing makes a sound man.

INTRODUCTION

TO THE

ELEMENTS OF EUCLID.

Euclid was born 300 years B.C. He lived at Alexandria, and wrote or compiled, in fact formed into one uniform whole, the treatise on geometry. known as *The Elements of Euclid.* It was written by him in Greek.

The work consists of thirteen books. The first six, with twenty-one propositions of the eleventh, and two from the twelfth, form the Euclid commonly read in English schools.

Euclid is looked upon as the father of geometrical reasoning, and so identified is his name with what he wrote about, that the word 'Euclid' in English schools is almost synonymous with 'geometry.'

The first six books of Euclid form a treatise on plane geometry. That is to say, all the lines you see on opening the pages of any Euclid, to the close of the sixth book, are supposed to be drawn on a plane or flat surface like the face of the table you write upon. The eleventh and twelfth books are about solid geometry, and refer to lines which might be drawn not only on the face of the table before you, but in any direction, either towards the ground or the ceiling or the walls of the room.

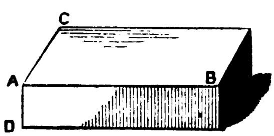
On Points, Lines, and Surfaces.

At present you have nothing to do with solid geometry, and yet it may help you the better to understand the definitions which Euclid gives of certain words which he is about to use, if we begin by considering a solid.

Every object which can be held in the hand has three dimensions, commonly called length, breadth, and thickness; and that which has length, breadth, and thickness, is called a solid. It matters not whether the object has a regular form like a brick, or an irregular form like a chance stone that might be picked up.

Let us now take in hand and consider a solid.

Here is a brick, we will say a wooden brick. Its length



measured from A to B is so many inches, that is one dimension. Its breadth measured from A to C is so many inches, that is a second dimension. Its thickness measured from A to D is so many inches, that is a third dimension.

Suppose now that its thickness be taken gradually away (planed away for instance) then the brick will become thinner and thinner; and when the thickness has been all taken away, there will remain only two dimensions, length and breadth (that is, in fact, the *face* of the brick). This is Euclid's superficies or surface.

After this, if the breadth of this surface be gradually taken away, the surface will become narrower and narrower, and when the breadth is all gone, one dimension only remains, that of length (the edge, in fact, of the brick). This is what Euclid calls a line. It has no thickness, no breadth, only length.

Lastly, suppose this line to shrivel up from one end; in other words suppose its length to be gradually taken away, the line will become shorter and shorter, till, when all the length is gone, there will remain no dimension at all; only

a dot, as it were, standing at the corner of the brick. Which dot has neither thickness, nor breadth, nor length. All the dimensions of the solid are gone; one after another they have been taken away. Now this dot without length, breadth, or thickness, is Euclid's point; it is called by him *σημεῖον*. And we are now prepared for his

Definition 1.—**A point is that which has no parts, or which has no magnitude.**

Some commentators on Euclid add,—but position.

Observe, *we* began with a solid brick, and taking it to pieces, resolving it,[1] that is, taking away successively its thickness, breadth, and length, we came at last to a *Point*.

Now Euclid takes the opposite course: he sets out with a point, and he seems to have before him the idea of the point moving. And as a pencil point moving along paper and leaving a trace behind, forms or generates a visible line, so his point, having no dimensions, no parts, no magnitude, by moving along, will form or generate a mathematical or Euclidian line, which he thus describes in

Definition 2.—**A line is length without breadth.**

He adds, what is suggested by the idea of the motion of a point generating a line,

Definition 3.—**The extremities of a line are points.**

He might have added, the intersection of two lines (that is, where they cross one another) is a point.

Now the motion of the point may be curved or crooked. The point may, in its motion, swerve first to one side and then to another, still the moving point (having itself no parts, no magnitude) will generate a line, that is, length without breadth. But if, in its motion, the point do not swerve one way or another, Euclid calls such a line a straight line, for he gives as

[1] Resolve, ἀνα-λύω, hence analysis.

> *Definition* 4.—A straight line is that which lies evenly between its extreme points.

This much Euclid says of lines, and then he proceeds to the next magnitude of which he speaks; namely, that which has length *and* breadth.

Here he seems to have before him the idea of the line, which has length only and no breadth, being drawn along in a direction different to its length (as the end of a rake is drawn by its handle), and so generating a surface which he thus defines:

> *Definition* 5.—A superficies (or surface) has only length and breadth.

He adds, what is suggested by this idea of a surface being generated by the motion of a line in a direction different to its length,

> *Definition* 6.—The extremities of a superficies (or surface) are lines.

In the seventh definition there is given a neat and satisfactory test of a flat surface. It is one naturally used by the polishers of marble surfaces. They take what they call a straight edge, and lay it in different directions on the surface which they are polishing. If the straight edge, wherever it is placed, exactly lies along the polished surface, so that no daylight is visible between the straight edge and the polished surface, then they know that the surface is truly flat. The actual words of the definition or test of a flat surface are these:

> *Definition* 7.—A plane superficies (in other words, a flat surface) is that in which any two points being taken, the straight line between them lies wholly in that superficies (or surface).

Of course if a flat surface were pushed in any direction except that of its length or breadth, it would generate a solid; in other words, the space through which the surface was pushed, if immediately crystallised, would be a solid

having the three dimensions of length, breadth, and thickness.

Thus beginning with a point, by the method of putting together,[1] that is, by adding successively length, breadth, and thickness, we get to a solid. Euclid, however, stops at surface, because the present portion of his treatise is on *plane* geometry.

We may now, then, suppose that we have a flat surface, defined as above, before us, on which to place points and to draw straight, or other lines.

We must here observe that the points which we place on the flat surface must have some magnitude, and the lines which we draw on it must have some thickness, else we should not see them. But it is not difficult to conceive the point to become finer and finer till at last it has lost all its magnitude. In like manner, we may conceive the line to become thinner and thinner, till it has lost all its thickness, and has only the dimension of length remaining. Thus the physical points and lines, for so they are called when they have magnitude and thickness—become mathematical points and lines according to Euclid's Definitions.

On Angles.

Take up your compass, and open it a little way. The amount of opening between the legs of the compass is an angle. As you open the compass more, the angle becomes greater, until the legs are drawn so far asunder that they are in a straight line; then there ceases to be (according to Euclid's definition) an angle between them. His definition of an angle being as follows:

> *Definition* 9.[2]—A plane rectilineal angle is the inclination of two straight lines which meet together, but are not in the same straight line.

The legs of the compass represent the two straight

[1] Putting together, συν-τίθημι, hence synthesis.

[2] Definition 8 is omitted; it refers to angles contained by curved lines, and is not required at present.

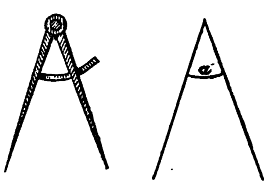
lines. They meet together at the hinge. When the legs are drawn so far apart that they are exactly in opposite directions, they meet together, but *are* in the same straight line.[1]

The forming an angle by drawing the legs of a pair of compasses asunder teaches incidentally what should be early learned and always remembered—that the magnitude of an angle does not depend on the length of the straight lines containing it, but solely on the amount of opening between them.

You may see as you look about you many examples of angles, the corner of the book before you is an angle, so is the corner of the table, or of the door, so again is the corner of a field where two hedge-rows meet. You may remember what Horace calls that cozy corner of his Tiburtine farm, where he hoped to sit in his old age, and drink his Falernian—

ANGULUS *ille mihi ridet.*

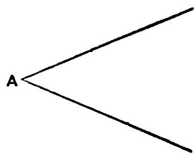
The compasses used by smiths and carpenters are sometimes made in the following manner. From one of the legs a curved rim projects, which moves freely through an opening in the other leg. In this leg there is a thumbscrew, by which the curved rim may be clamped. The legs may be thus kept at any desired distance from each other.



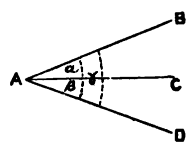
This form of compasses suggests a method of indicating an angle, very commonly used in books on trigonometry, namely, that of putting a dotted curved line between the straight lines which contain the angle. A letter (generally a Greek one, as α) is often inserted in the opening; and the angle between the straight lines is called the angle α.

[1] Hence the following definition of an angle has been suggested: An angle is the difference of direction of two straight lines which meet.

But the learner must not be misled by this figure, to imagine that the angle is the shut-up space, within which the letter α is placed. The dotted line is put there only to draw attention to the amount of opening between the straight lines.



When only two straight lines meet at a point, Euclid indicates the angle between them by a letter (as A) placed at the point of meeting of the two straight lines, and the angle between the straight lines he calls the angle A.



If more than two straight lines should meet at a point the single letter A would fail to tell whether it was intended to indicate the angle α or the angle β, or indeed the larger angle γ, made up of the angles α and β together.

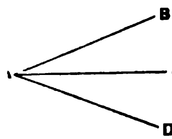
In this case Euclid uses three letters to indicate the angle. The angle α, that is the opening between the straight lines B A, A C, he indicates by the letters B A C or C A B. The angle β, that is the opening between the straight lines C A, A D, he indicates by the letters C A D, or D A C; and the angle γ, that is the opening between the straight lines B A, A D, he indicates by the letters B A D, or D A B, the letter at the angular point being always the middle letter of the three.

This latter figure will help the learner to understand the adding angles together, or the subtracting one angle from another. The angle α, which is indicated by the letters B A C, may be added to the angle β, indicated by the letters C A D, and taken together they make up tho angle γ, indicated by the letters B A D.

So the angle β (that is C A D) may be taken from the angle γ (that is B A D), and the remainder will be the angle α (that is B A C).

It may be well here to put the learner on his guard against an error, a very odd and unexpected one, some-

times made. When a teacher says to a learner:—'If from the angle B A D you take the angle C A D, what angle remains?' it is not uncommon for him to reply, 'If you take away the angle C A D, all that remains is the line A B.'

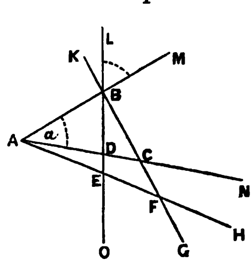


Now this is an entire misapprehension. Angles and lines are different things altogether. If an angle is taken from an angle, it is *not* a line that remains, but an angle, just as when you take shillings from shillings it must be shillings, and not inches or ounces, that are left.

In fact, the intermediate line C A belongs, so to speak, to both angles: when taken with A B, it makes the angle C A B (or α), and when taken with A D it makes the angle C A D (or β). And when either of the two, as α, is taken from the whole angle γ, it is the other angle β that remains. In other words; if from the angle B A D we take the angle B A C, the angle C A D remains.

As it is of the highest importance to see clearly and quickly the angle which is contained between any two straight lines which meet, it would be well for the learner to draw some interlacing straight lines, and to examine himself, or to get some friend to examine him in pointing out the angle contained by any two named lines which meet; or, the angle being pointed out, in naming the straight lines which contain it.

For example. If the friend says, 'What is the angle contained by L B, B M?' let the learner place a little curved line between these straight lines, as in figure, to mark the angle; or if the friend puts the curved line between any two straight lines which meet, let the learner name to him the two straight lines which contain the angle so marked.

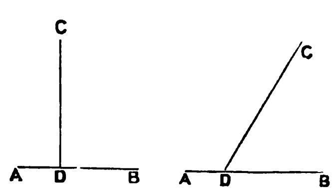


In naming the straight lines containing the angle pointed out, let the learner be advised to take the first line named *towards* the angular point, and the second *from* the angular point. For instance, if he wishes to name the straight lines containing the angle marked a, let him say that the angle is contained by the straight lines B A, A D, in preference to saying that it is contained by the straight lines A B, A D, or by B A, D A. This may be thought an immaterial remark, but it is found that learners who name the lines in the way here recommended, learn most quickly to distinguish the angle contained between two named straight lines.

After one more remark in reference to angles, we will pass to the next definition. It will be seen (last fig.) that the angle contained by B A, A D, by B A, A C, or indeed by M A, A N, is one and the same angle—the one marked in the figure as the angle a. This teaches that it matters not at what points in the lines containing an angle we place the letters which indicate it.

On Right Angles, Obtuse Angles, and Acute Angles.

Suppose A B to represent the edge of your desk or table, and C D your ruler. If you bring your ruler, C D, to the edge of your desk, A B, and hold it there, on end, in such a way that the angles C D A and C D B [these are called adjacent angles] are equal, each of the two is a right angle.



If you hold the ruler so as to make one of the adjacent angles greater than the other, then the greater angle, C D A, is called an obtuse angle (from *obtusus*, blunt), and the lesser, C D B, is called an acute angle (from *acutus*, sharp). This illustration leads on to the Definitions 10, 11, and 12 of Euclid, which are as follows:

Definition 10.—When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.

Definition 11.—An obtuse angle is that which is greater than a right angle.

Definition 12.—An acute angle is that which is less than a right angle.

On the Circle.

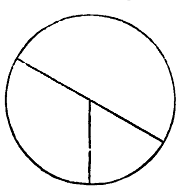
Euclid now passes on to a new kind of magnitude, which is introduced by the two following definitions:

Definition 13.—A term or boundary is the extremity of anything.

Definition 14.—A figure is that which is enclosed by one or more boundaries.

The 'figure' so introduced is the Circle, which is thus defined:

Definition 15.—A circle is a plane [or flat] figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within it to the circumference are equal to one another.



He adds:

Definition 16.—And this point is called the centre of the circle.

A primitive mode then of describing a circle would be

to fix a pin through your paper into a drawing board, and attaching the point of a pencil to the pin by a thread, to draw the pencil point round, keeping the thread always tight.

A much more convenient method, however, is by means of a pair of compasses, by which the distance of the circumference from the centre can be regulated at pleasure.

The definition of a circle should be remembered and repeated with precision. Frequently in repeating the definition the word 'straight' is left out, also it is a not uncommon mistake to omit the words 'to the circumference,' both which omissions spoil the precision and accuracy of the definition.

Euclid adds three more definitions about the circle. First he defines a diameter thus:

> *Definition* 17.—The diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

To this might be added: A radius is a straight line drawn from the centre of a circle to the circumference. The two next definitions, the 18th and 19th, may be passed over here, as they are not required for the first twelve propositions.

On Rectilineal Figures, especially Triangles.

With regard to rectilineal figures, Euclid says:

> *Definition* 20.—Rectilineal figures are those which are contained by straight lines.

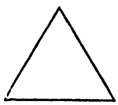
Such rectilineal figures he divides into three classes, thus:

> *Definition* 21.—Trilateral figures or triangles are those which are contained by three straight lines.

> *Definition* 22.—Quadrilateral figures are those which are contained by four straight lines.

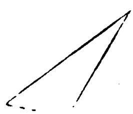
Definition 23.—Multilateral figures (from *multus*, many; *latus*, side) or polygons (from πολύς, many; γωνία, angle) are those which are contained by more than four straight lines.

In the next three definitions Euclid divides triangles into three classes *according to their sides.*



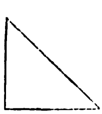
Definition 24.—If the three sides are equal, the triangle is equilateral (from *æquus*, equal; *latus*, side).

Definition 25.—If two sides only are equal, the triangle is isosceles (from ἴσος, equal; σκέλος, leg).

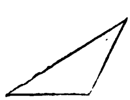


Definition 26.—If all the three sides are unequal, the triangle is scalene (from σκαληνός, unequal).

He then divides triangles into three classes *according to their angles*, thus:

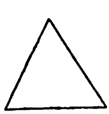


Definition 27.—If one of the angles is a right angle, the triangle is a right-angled triangle.



Definition 28.—If one of the angles is an obtuse angle, the triangle is an obtuse-angled triangle.

[It will be seen hereafter that a triangle can only have one right angle, or one obtuse angle.]

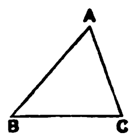


Definition 29.—If all the angles are acute, the triangle is an acute-angled triangle.

Here Euclid closes his definitions about triangles; but

as trilateral figures, or triangles, will meet us at every step of our journey through Euclid, it may be well here to make two or three observations about them which may be helpful to beginners.

Obs. 1.—If you indicate an *angle* by using three letters you must be careful, as was before said, to put the letter standing at the angular point between the other two. But if you indicate a *triangle* by using three letters, you may take the letters in any order you like. Thus, the triangle represented in the annexed figure may be called the triangle A B C, or the triangle C B A, or C A B, &c.



Obs. 2.—When you have a triangle before you as A B C in the last figure, you may consider any one of the three straight lines which form it to be the base on which it stands, and then the two remaining straight lines are called its *two sides.* Thus if B C be considered the base, the *two sides* are B A, A C. The *vertex* is the point A, where the two sides meet. The angle B A C is *the angle at the vertex.* This angle may also be distinguished as *the angle opposite to the base*, or as *the angle contained by the two sides.* The two other angles A B C, A C B, are called *the angles at the base*, and it is to be particularly noticed that each of them is opposite to one of the two sides; viz., the angle A B C is opposite to the side A C, and the angle A C B is opposite to the side A B.

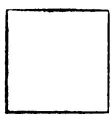
As an exercise, it may be well for a learner to take A B, and then A C as the base, and to write down in each of these cases which lines will be the two sides, which point will be the vertex, which angle will be the angle at the vertex, distinguished also as the angle contained by the two sides, or as the angle opposite to the base; and which will be the angles at the base, and to what sides they will be opposite.

Obs. 3.—A triangle is a figure bounded by three straight lines; and consequently it has three angles. It also may be considered as a quantity of surface or area enclosed by

three straight lines. And when it is said of two triangles that they are equal in every respect, it is meant that the three sides are equal each to each :—that the three angles are equal each to each :—and that the enclosed surfaces or areas are equal. In fact, two triangles equal in every respect, are equal in respect of *sides*, of *angles*, and of *area*.

A learner will find it useful to remember this remark when he is doing the fourth Proposition.

Of four-sided figures the most important is a square, which Euclid thus defines :



Definition 30.—A square is that which has all its sides equal and all its angles right angles.

The remaining definitions of four-sided figures are not essential for our present purpose; we will therefore omit them, as well as all reference to parallel lines.

On the Postulates.

At the close of the Definitions Euclid inserts three Postulates, that is, three things which he asks to be allowed to do (from *postulare*, to ask).

He says, Let it be granted,

Postulate 1.—That a straight line may be drawn from any one point to any other point :

Postulate 2.—That a terminated straight line may be produced to any length in a straight line :

Postulate 3.—That a circle may be described from any centre at any distance from that centre.

In other words, he asks to be allowed the use of a ruler, and a pair of compasses ; not to be used for measuring with, but only in order to draw straight lines, to produce (i.e. to lengthen) terminated straight lines, and to describe circles.

It will be enough for a beginner if he takes in this meaning of the postulates, though doubtless they bear a deeper meaning than this.

It is impossible to draw a line mathematically straight. It requires a perfectly straight edge, and a perfectly steady hand, neither of which are attainable. The difficulty, or rather impossibility, of producing a terminated straight line, is a proverb among draughtsmen. And to describe a perfect circle, the centre must be a mathematical point, not to speak of other causes of imperfection. Now in the postulates, Euclid asks that it may be granted that his straight lines and circles may be in accordance with his definitions, and ideally correct, that is, free from any imperfection arising either from imperfect instruments or imperfect drawing.

On the Axioms.

We pass on now to the Axioms, that is, to certain propositions or statements which Euclid asks may be taken as true without proof (from ἀξιόω, I claim).

Observe: He does not ask that they may be taken as true because they do not *need* proof, but because he has no proof to offer. He proves whatever he can, however self-evident it may be. It is his wish to reduce the number of axioms to the smallest possible.

The first axiom is as follows:

> *Axiom* 1.—Things which are equal to the same thing are equal to one another.

Now this axiom, simple as it sounds, is often, very often, incorrectly applied. The following is an incorrect application of it.

Suppose we have three things, we will call them A, B, and C; and suppose that we know that A is equal to B, and that C is also equal to B. The following is an incorrect application of the axiom. The incorrect conclusion is printed in italics.

We have here A and C, each of them equal to B; and things which are equal to the same thing are equal to one another.

Therefore A, B, and C are all equal to one another (Ax. 1).

This is wrong; the conclusion of the argument ought to have been:

Therefore A is equal to C.

As soon as A has been thus proved equal to C by the first axiom, A, B, and C may be said to be all equal to one another, but not before.

The next axioms are as follows:—

> *Axiom* 2.—If equals be added to equals the wholes are equal.
>
> *Axiom* 3.—If equals be taken from equals the remainders are equal.
>
> *Axiom* 4.—If equals be added to unequals, the wholes are unequal.
>
> *Axiom* 5.—If equals be taken from unequals the remainders are unequal.
>
> *Axiom* 6.—Things which are double of the same thing are equal to one another.[1]
>
> *Axiom* 7.—Things which are halves of the same thing are equal to one another.[1]

The eighth Axiom is as follows:

> *Axiom* 8.—Magnitudes which exactly fill the same space are equal to one another.

This is a very important axiom; a great deal depends upon it. Observe Euclid does not say magnitudes are equal which fill equal spaces, but which fill the same, the very same space.

As an illustration of his meaning, suppose you are casting bullets with a mould. There are two bullets before you which you have cast in the same mould; they are equal in accordance with the eighth axiom, for they have

[1] On reflection it will be seen that axiom 6 is a particular case of axiom 2, and that axiom 7 is a particular case of axiom 3.

both filled the same space, that is, the hollow of the mould in which they were cast.

In the fourth Proposition great use is made of this axiom. Under certain conditions, there named, Euclid proves that two triangles are equal in every respect (see *Obs.* 3 on triangles, p. 13). Now before he draws the conclusion that the lines, the angles, and the areas required to be proved equal,—*are equal*, you will find that he will be most careful to show that the lines, the angles, and the areas in question may be made, severally, *to fill the same identical spaces.*

We now pass on;

Axiom 9.—The whole is greater than its part; or, than a part of it.

Axiom 10.—Two straight lines cannot enclose a space.

It takes *at least* three straight lines to inclose *or shut up* a space. If, then, two lines enclose a space, we know that one or both of the lines are not straight, for two straight lines *cannot* enclose a space.

Axiom 11.—All right angles are equal to one another.

Modern geometricians have tried to remove this eleventh axiom from the list of axioms, and to give a proof that all right angles are equal; hereafter you may examine such proofs.

The twelfth axiom is not required till the twenty-ninth proposition has been reached, and therefore no allusion to it is required in the present treatise.

GENERAL REMARKS ON THE PROPOSITIONS.

In every proposition some thing is given as a foundation to start from; for instance, in the first proposition a straight line of a certain length is given; in the fifth proposition a triangle having two equal sides is given. In some of the propositions, starting from the given foundation (or, as it is commonly called, 'hypothesis,' from ὑπο-τίθημι, to lay down as a foundation), you are required to *do* something; for instance, in the first proposition, on the straight line of given length, you are required to describe an equilateral triangle. In other propositions, you are required, from the given hypothesis, to *prove* something; for instance, in the fifth you are required to prove that the angles at the base of the equal-sided (or isosceles) triangle are equal.

The first kind of proposition, in which you have to *do* something, is called a PROBLEM (from προβάλλω, to set before one); the second kind, in which you have to *prove* something, is called a THEOREM (from θεωρέω, to consider). These words are printed in Euclid at the head of each proposition.

It is of the highest importance that a beginner should see clearly in every proposition what it is that is given, and what he has to do or to prove, as the case may be. And no better advice can be given to a learner, who wishes to understand and to profit by Euclid, than this:—Never to go farther than the heading (or enunciation) of a proposition until he sees clearly what is given, and what has to be done, if it is a problem,—or what has to be proved, if it is a theorem.

Indeed, so important is it that a learner should see this clearly, that in modern editions of Euclid the enunciations (or headings in which this is stated) are often divided into

two paragraphs, the first stating what is given, and the second what has to be done or has to be proved. In the following pages this is done to a greater extent than has been thought necessary in other editions of Euclid.

Observe, further, that each proposition consists of several distinct portions.

First, there is the heading, or general enunciation; that is, the statement of what is given and what has to be done or to be proved, in *general* terms, without reference to any particular lines or figure.

Then follows the particular enunciation. That is, the previous general enunciation is repeated, but now with reference to, and explained by, the lines and figure actually given and drawn.

Then follows, commonly, the construction, in which certain lines, which are necessary in order to do what has to be done, or to prove what has to be proved, are drawn. Sometimes, but very seldom, no construction is required.

Lastly follows the demonstration or proof. In this it is shown (mostly by the help of the lines of construction) that what was to be done *has been done*, or what was to be proved *has been proved.*

The learner is advised to have before him two separate pieces of paper, or two pages of an exercise book, on one to draw the figure, and on the other to write each paragraph, one by one, as soon as its meaning is understood.

Great deliberation is strongly recommended on beginning Euclid. We teachers of Euclid are often much to blame for the quick, off-hand way in which we take pupils through the early propositions. They seem so simple to us, we are so familiar with the ideas and processes of Euclid, that without intending it, we often dazzle, bewilder, confuse our pupils by starting at too quick a pace.

A story which the writer heard in early life from a relative, of the good old days of mail-coaches and posting,

has been so useful to himself in teaching Euclid, that he will here repeat it for the benefit of others.

His relative was bound for Ireland; he wished to catch the Holyhead mail. On driving up to the inn-door where the mail changed horses, he found to his dismay that the coach was gone; it had started some minutes. An old wiry post-boy, standing by, said to him, 'I'll catch it.' 'Put your horses to,' was the order. In an incredibly short time the horses are harnessed, and the light, yellow-panneled post-chaise is brought out of the yard. He gets in and they are off.

The impatient traveller was dismayed at finding that the old post-boy started at a deliberate trot, not by any means at the pace of the mail they had to catch, and leaning eagerly forward with his head and shoulders out of the front-window he cried, again and again, 'Drive on, my man, drive on; you'll never catch the mail.' The only answer of the post-boy was a dig back of the right elbow, as much as to say, 'I know what I'm about.'

The traveller, finding the obstinacy of the post-boy immovable, fell back on his seat in despair. But by-and-by he feels that the trot is quickening. At a touch of the post-boy's whip on each shoulder, the horses spring into a sharp canter, which becomes a gallop soon; and on they go, now at a full gallop up and down hill, the post-chaise swaying as they speed along. The Holyhead mail is in sight, they dash up along-side; the coachman pulls up for a moment; down comes the guard, and while they are hurrying the passenger into the mail, the old post-boy, pocketing his fee, said with a grin, 'I told you I'd catch it. You'd have had me blow my hosses at startin', and then I'd never ha' done it.'

And so he left the passenger to ruminate on the occurrence, and from it to learn a lesson for life. It is here given for the benefit of those who, whether as teachers or learners, are engaged on the first pages of Euclid.

We will now quietly and deliberately take in hand the first proposition.

THE FIRST PROPOSITION DISCUSSED.

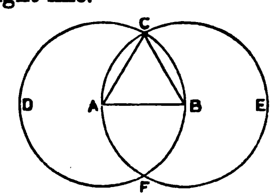
The general enunciation is as follows :—

Given a straight line :—

It is required to describe an equilateral triangle upon it.

Supposing the learner to have before him the two pieces of paper recommended on page 19. He should write on one of them the first clause of the particular enunciation, thus :—

'1. Let A B be the given straight line.' [1]



And on the other piece of paper he should draw a straight line, and put A at one extremity of it and B at the other. These letters will indicate the extreme points of the line. The extremities of a line are points (Def. 3).

The learner has thus the given straight line before him, and, keeping his eye on the general enunciation, he can complete the particular enunciation by writing thus :—

'2. It is required to describe an equilateral triangle upon A B' (merely putting A B for the word *it* in the general enunciation).

Then follows the construction : [2]

'1. From the centre A, and at the distance A B, describe the circle B C D.'

[1] The pupil will notice in the discussion of the first and following propositions, that he is to write what is included between the inverted commas (' '). What is not so included is to be read, but not written.

[2] In the course of the construction and demonstration of the first proposition, reference is made to the first and third postulates, to the fifteenth definition, and to the first axiom. And similarly in the subsequent propositions. Some editors strongly recommend that a beginner should repeat, or write out, such references before doing the proposition.

The beginner will remember that this, which he is here told to do, is one of the three things which Euclid asks to be allowed to do, namely in Postulate 3.

Let him then do it; placing the steel point of his compass at the extremity A of the straight line A B, let him bring the end of its pencil leg to the point B, and sweep round the circle B C D.

The second step in the construction is,

'2. From the centre B, and at the distance B A, describe the circle A C E.'

Let the learner describe this circle as he did the former; placing the steel point of his compass at the extremity B, and the end of the pencil leg at the extremity A, let him sweep round the circle A C E.

The pupil will notice that the letter C is placed where the two circles cut each other above the straight line A B. The letters D and E may be placed at any other points in the circumferences of the first and second circle.

We now pass to the third step of the construction,

'3. From the point C where the circles cut one another, draw the straight lines C A, C B, to the points A and B.'

This again it will be remembered is one of the three things which Euclid asks to be allowed to do, namely in Postulate 1. Let the learner actually draw the straight lines C A and C B.

The beginner is advised to draw these carefully with ruler and pencil.[1] Place the ruler so that the point of the pencil shall come exactly upon the intersection C of the two circles, and also exactly on the extremity A of the given line. Then draw the line carefully from point to point. In the same careful manner draw the straight line C B. Complete this portion of the proposition by writing the words:

'4. Then A B C shall be an equilateral triangle.'

[1] Those who wish to make the figures very neatly should first draw them with a fine pencil, and afterwards ink in the lines with Indian ink, using for this purpose the pen-leg of the compass, or a drawing-pen. What is *given* should always be 'inked in.'

There remains the demonstration, that the triangle A B C which is here *said* to be equilateral, is really so. Let it be thus written:

'1. A C is equal to A B, because they are both radii of the same circle B C D (Def. 15).'

If the learner looks back to Definition 15, he will there see that all straight lines drawn from the centre to the circumference of the same circle are equal. In other words, that all radii of the same circle are equal.

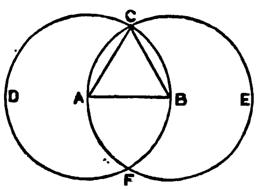
'2. C B is equal to A B because they are both radii of the circle A C E (Def. 15).

'3. Wherefore A C and C B are each equal to A B.'

Now look back to Axiom 1, and introduce it into the demonstration by writing thus:

'And things which are equal to the same thing are equal to one another.'

[Do not write as the conclusion, Therefore C A, A B, and B C, are all equal to one another; that would be wrong (see obs. on Axiom 1, page 15). The right conclusion is,]



'4. Therefore C A is equal to C B, ax. 1.'

Now, indeed, that you have shown

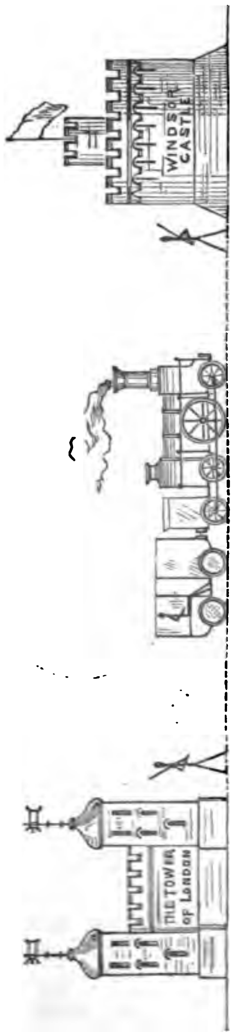
(1) that A C is equal to A B,
(2) that C B is equal to A B,
and (3) that A C is equal to C B:

That is—now that you have shown that each side of the triangle A B C is equal to each of the other two—you *can* say, Therefore C A, A B, and B C are all equal to one another. Write then as follows:

'5. Therefore C A, A B, and B C are all equal to one another.'

And add in conclusion:

'Wherefore the triangle A B C is equilateral, and it is described on the straight line A B, which was required to be done.' The First Proposition is thus completed.



As very many learners fall into the mistake of wrongly applying the first axiom in the above and the two following propositions, suppose we try to show how to apply it rightly by a jocular illustration. Horace says

> Ridiculum acri
> Fortius et melius magnas plerumque secat res.

On the left of the adjoining woodcut is drawn a turreted fortress, that is meant to represent the Tower of London; on the right is drawn a castle with a round tower and a flag flying, that represents Windsor Castle.

Now the first battalion of the Grenadier Guards is stationed at the Tower, and the second is stationed at Windsor. And you here see two Privates of these battalions pacing backwards and forwards as sentries, one at the Tower, and the other at Windsor Castle.

Between these two sentries a bet, a heavy and important bet, is pending, as to which of them is the taller.

Unfortunately, as you perceive, they are both on guard, and cannot therefore decide the question by the simple mode of standing back to back and measuring.

What shall they do? The London man has a friend who measures

with him, and finds that his height is exactly that of the sentry.

Now this friend is not on duty, and he gets leave to go to Windsor. In the middle of the picture you see him in the train on his way. When he gets there he measures with the man that is on guard at Windsor Castle, and he finds that his height is exactly that of this sentry also.

Hence the London man and the Windsor man know that they are the same height the one as the other, *though they have not been able to measure the one with the other.*

Remember the important question to be decided is this, whether the London man is equal to the Windsor man. The man that went by the train is only useful as a means of comparing these two.

As soon then as you have employed the man that went by the train to serve your purpose of comparing the London and the Windsor man, cast him aside, put him out of sight, and fix your mind on the important personages, the London man and the Windsor man, who are now known to be equal to each other, because they are each equal to the man that went by the train.

Look now at the figure in Prop. 1, and say of the three straight lines which form the triangle A B C, which two lines represent the London man and the Windsor man, and which line represents the man that went by the train.

[The learner is advised here to close the book and write the answer to this question, and when he has written it, to read on and see if he is right.]

Ans.—C A is the London man, C B the Windsor man, and A B is the man that went by the train.

As in the foregoing discussion of the First Proposition, the thread of the reasoning was often broken off by explanatory remarks, before passing on to the next proposition it may be well to write it out without explanatory remarks, as it might be written in an examination.

FIRST PROPOSITION WRITTEN OUT.

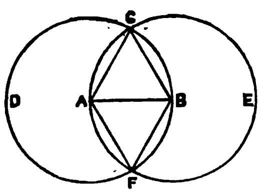
Proposition I. Problem.

General Enunciation.

Given a finite straight line:—

It is required to describe an equilateral triangle upon it.

Particular Enunciation.



Let A B be a finite straight line:—

It is required to describe an equilateral triangle upon A B.

Construction.

1. From the centre A and at the distance A B describe the circle B C D (Post. 3).

2. From the centre B and at the distance B A describe the circle A C E (Post. 3).

3. From the point C where these circles cut one another draw the straight lines C A, C B, to the points A and B.

4. Then A B C shall be the equilateral triangle required.

Demonstration.

1. A C is equal to A B, because they are radii of the same circle B C D (Def. 15).

2. C B is equal to A B, because they are radii of the same circle A C E (Def. 15).

3. Wherefore A C and C B are each of them equal to A B.

4. And things which are equal to the same thing are equal to one another. Therefore A C is equal to C B (Ax. 1).

5. Wherefore C A, A B, and B C are all equal to one another, and the triangle A B C is therefore equilateral, and it is described on the given straight line A B, which was to be done.[1]

Observe the enunciation, construction, and demonstration of the above proposition all end in the same way.

1. The enunciation ends thus :—It is required on A B to describe an equilateral triangle.

2. The construction ends thus :—Then shall A B C be the equilateral triangle required.

3. The demonstration ends thus :—The triangle A B C is therefore equilateral, and it is described on the given straight line A B, which was to be done.

Let the young geometrician look out for a similar correspondence in the close of the enunciation, construction, and demonstration in all *Problems* he may have to write out.

And let him remember that no Proposition, whether problem or theorem is properly written out, unless the close of the demonstration corresponds to that of the enunciation.

One word more : Let all learners read this treatise with 'pen in hand' ; and always construct (build up, see p. 29) the figure according to the directions given, on a loose bit of paper. And having done this, let them *use their own figure, not the one given in the book.* Whoever does not attend to this rule will find himself often distracted and hampered by having to look back to the figure on a previous page.

[1] It will be a good exercise to put the letter F at the other point where the circles intersect, and completing the triangle A B F as above, to prove that the triangle A B F is equilateral, in a similar manner.

THE SECOND PROPOSITION DISCUSSED.

We now pass on to the second proposition.

You will find that it depends on (hangs on) the first. Here, then, you have a first instance of what you will observe constantly in your progress through Euclid—the dependence of successive propositions on those which have gone before. As you advance from proposition to proposition, you will find a following one so often linked to and dependent on the next preceding, as to bring to mind the idea of a chain in which the successive links are joined on to, and hang, one from another.

Indeed, it is instructive in going through the propositions of Euclid to hang them, chain-wise, one from the other, linking each to that on which it depends, as is done in the woodcut facing the title-page, which will explain itself to you as you proceed. In fact, in the second proposition, you will find that you join on the second link of your 'chain of reasoning.' [1]

The question ever to be asked, what is given, and what has to be done in the Second Proposition, is thus answered in the

General Enunciation.

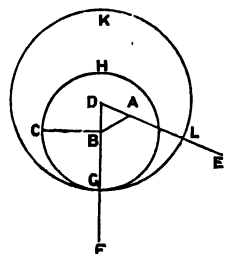
Given a point, and a straight line:—

It is required from the given point to draw a straight line equal to the given straight line.

Now put a dot where you like to indicate the place

[1] The learner is advised, as he finishes each successive proposition, to turn to the frontispiece, and observe how the corresponding link is hung on the chain. A full explanation of the woodcut is given on page 124.

of the given point. You had better put it about the middle of your sheet of paper, and call the point A. Also draw a straight line where you like, to indicate the given straight line (let one of its extremities be about an inch from A). Put the letters B and C, one at each end of the straight line, and call it the straight line B C.



This being done, looking back to, or remembering the general enunciation, write as follows the

Particular Enunciation.

'Let A be the given point, and let B C be the given straight line :—'

'It is required from A to draw a straight line equal to B C.'

Before the learner begins the construction, he would do well to listen to the following word of advice: it applies to every proposition:—Avoid always the miserable habit of copying at once the whole figure as given in the book. The figure ought to be constructed, built up (*construere* means to build) gradually, according to the directions given in the successive steps of the construction.

We will now go on to the

Construction.[1]

Write: '(1) From the point A to B, draw the straight line A B.'

Then do this carefully with pencil and ruler, observing the directions for doing it given in Prop. I. Remember it is done by Postulate 1.

This being done, write:

'2. Upon A B describe an equilateral triangle D A B.

[1] See note 2 on Prop. I., p. 21.

Now this is to be done by the first proposition. Euclid, however, does not encumber his figure by exhibiting the construction of a previous proposition, on which a following one depends. But in order to construct the figure neatly and correctly, you should draw lightly in pencil, as in Prop. I., two circles with the centres A and B respectively, and with the radius A B, intersecting at the point D. Having thus found D the vertex of the equilateral triangle, the straight lines D A, D B may be drawn to the points A and B, and inked in, and when the ink is dry, the two circles drawn in pencil may be effaced, and you have the equilateral triangle as it appears in Euclid's figure of the second proposition.

Another way, in common use among draughtsmen, is to open your compasses so that the distance between the points may be equal to A B. Then with the centres A and B, draw two little arcs, that is, small portions of the circumference, about where you judge the circles will intersect. The intersection of these arcs gives the point D, that is, the vertex of the equilateral triangle, which may be completed, as before, by joining A D and D B.

The equilateral triangle being thus described, write down the next step of the construction, which is:

'3. Produce the two sides D A, D B to the points E and F.' This is done by Postulate 2.

Notice that both the straight lines are produced from the same point D, through A and B.

The producing a straight line is not very easy to do accurately; it requires care. The ruler must be placed very carefully along the straight line to be produced, and before producing it the learner must observe if the point of the pencil, when it is drawn along the ruler, passes accurately from end to end, along the straight line to be produced. The straight lines D A, D B may be produced as far as you like, to the edge of your paper if you like.

Having produced the straight lines D A, D B to E and F, write:

'4. With centre B, and at the distance B C, describe the circle C G H.'

Then describe it, as you were taught to do it, in Prop. I. At the point where this circle cuts the line B F, put the letter G. Also put H at any other point in the circumference.

Then write:

'5. With the centre D, and at the distance D G, describe the circle G K L.' [Observe that D is the vertex of the equilateral triangle, and that the distance D G is determined by the first circle C G H.]

Then do so. Put L at the point where this circle cuts the straight line D E, and K at any other point in the circumference of the circle. And add:—

'6. Then shall A L be equal to B C.'

The demonstration that it is so may be written as follows:

'1. B C is equal to B G, because they are both radii of the same circle, C G H (Def. 15).'

'2. Also D G is equal to D L, because they are both radii of the circle, G L K (Def. 15).'

[Now observe, D A is a part of the straight line D L, and if D A be taken from D L, the remainder will be A L.

Also D B is a part of D G, and if D B be taken from D G, the remainder will be B G.

We know, too, that D A and D B are equal, because they are the sides of an equilateral triangle.

Also we know that if equals be taken from equals, the remainders are equal (Ax. 3).]

We may then continue the demonstration from step 2, above, by writing thus:

'3. And D A and D B, parts of them, are equal, because they are the sides of an equilateral triangle.'

'4. Therefore the remainder A L is equal to the remainder B G (Ax. 3).'

'5. But B C was proved equal to B G.'

'6. Therefore A L and B C are each of them equal to B G.'

'7. And things which are equal to the same thing are equal to one another.'

[Do not write, therefore, A L, B G, B C are equal to one another, but write,]

'Therefore, A L is equal to B C (Ax. 1).'

'Wherefore from the given point A a straight line A L has been drawn equal to B C, which was to be done.'

Having now completed the second proposition, the learner will do well to try if he can detect which two lines are the London and Windsor man, and which line is the man that went by the train. [Having written the answer, let him read on and see if he is right.]

Ans.—A L and C B are the Windsor man and London man respectively, and B G the man that went by the train.

Obs.—It will be a great help in writing out the demonstration of every proposition to keep in mind the following remark:—

Whatever condition is given in the enunciation, and whatever you are told to do in the construction, is sure to follow the word 'because' in the demonstration.

For instance, in the construction of this proposition you were told, with centre B, and at the distance B C, to describe a circle C G H; and in the demonstration you meet the sentence, 'B C is equal to B G, because they are both radii of the same circle C G H.'

Again, you were told in the construction, with the centre D, and at the distance D G, to describe the circle G L K; and in the demonstration you have to say, 'D L is equal to D G, because they are both radii of the same circle G L K.'

Once more: In the construction you were told on A B to describe an equilateral triangle D A B; and in the demonstration you have to say, 'D A is equal to D B, because they re the sides of an equilateral triangle.'

This is an invariable, essential rule in geometrical reasoning, and it is very helpful to a beginner, when he has taken it in.

It remains now, after the foregoing discussion, to write out the second proposition without explanatory remarks, as it might be written out in an examination.

SECOND PROPOSITION WRITTEN OUT.

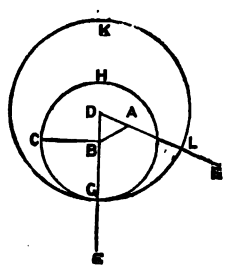
Proposition II. Problem.

General Enunciation.

Given a point and a straight line:—

It is required from the given point to draw a straight line equal to the given straight line.

Particular Enunciation.



Let A be a given point, and B C a given straight line:—

It is required to draw from A a straight line equal to B C.

Construction.

1. From the point A to B draw the straight line A B (Post. 1).

2. On A B describe an equilateral triangle D A B (Prop. 1).

3. Produce the straight lines D A, D B, to any distances E and F (Post. 2).

4. From the centre B, and at the distance B C

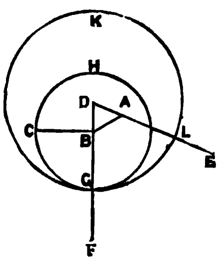
describe the circle C G H (Post. 3), cutting D F in the point G.

5. From the centre D, and at the distance D G, describe the circle G K L, cutting D E in the point L.

Then shall A L be equal to B C.

Demonstration.

1. B C is equal to B G, because they are both radii of the circle C G H (Def. 15).



2. D G is equal to D L, because they are both radii of the circle G L K (Def. 15).

3. D A, D B, parts of them, are equal, because they are sides of an equilateral triangle.

4. Therefore[1] the remainders B G and A L are equal (Ax. 3).

5. But B C was proved equal to B G.

6. Therefore B C and A L are each of them equal to B G.

7. And if two things are each equal to the same thing, they are equal to one another.

Therefore A L is equal to B C (Ax. 1).

Wherefore[1] from the given point A, a straight line A L has been drawn equal to the given straight line B C, which was to be done.

Observe here the connection between the endings of the enunciation, construction, and demonstration, which you were called to notice in Prop. I. (see p. 27).

[1] A few words on the distinction between 'therefore' and 'where fore as used in Euclid are given on page 42.

On Varieties in the Figure of the second Proposition.

Learners will find it useful to write out the construction of the second proposition without using letters of reference, in some such way as the following:—

1. Join the given point with one end of the given straight line. [Call this the joining line.]

2. Upon this joining line as the base, describe an equilateral triangle.

3. Produce the two sides of this equilateral triangle.

4. Taking the end of the given straight line which has been joined to the given point as the centre, and the length of the given straight line as radius, describe a circle.

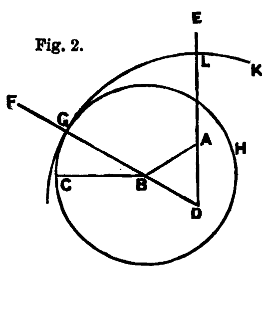
5. This circle will intercept, between its centre and circumference, a part of the straight line produced through the end of the given straight line, equal to the given straight line. [illegible]

6. Then with the vertex of the equilateral triangle, as centre, and with a straight line made up of this intercepted part, and the side of the equilateral triangle, as radius, describe a second circle.

7. The part of the straight line produced through the given point, which is intercepted between the given point and the circumference of this second circle, will be the straight line required.

Keeping in mind the above mode of writing the construction of Proposition II. it will be found an instructive exercise to make, besides the figure given on page 33, a second, third, and fourth figure to Prop. 2.

Fig. 2.



The second by describing the equilateral triangle on the other side of A B.

The third by joining A with C the other extremity of the given straight line BC, and describing the equilateral triangle on the upper side of the joining line A C. The fourth by describing the equilateral triangle on the other side of the joining line A C.[1]

Fig. 3.

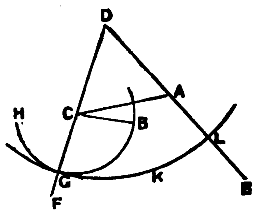
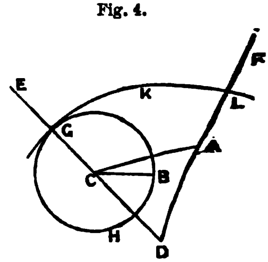
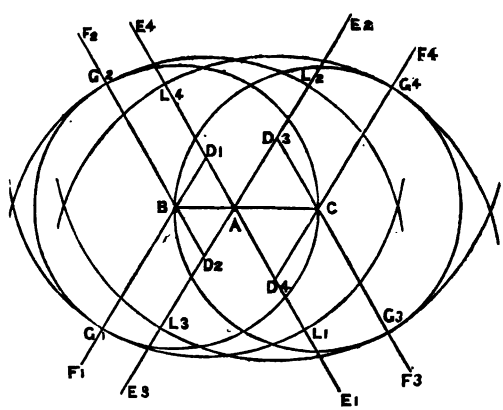


Fig. 4.



It will be a useful exercise, also, to place the given point somewhere in the given straight line B C.



It will not be necessary, in this case, to join A to B or C, because these points are already joined (A B and A C being parts of the given straight line B C); and from A, as

[1] These four varieties of the figure of the second Proposition may also be drawn in one figure, as is done in the next exercise, where a new variety of the figure is suggested.

before, a straight line, indeed four straight lines, $A L_1$, $A L_2$, $A L_3$, $A L_4$, may be drawn, each equal to B C.[1]

Observe the nearer the given point is placed to the end of the given straight line, the smaller becomes the side of the equilateral triangle, and when the given point A coincides with the end of the given straight line, the equilateral triangle becomes reduced to a point, and the two circles become one and the same, the centres of both being the same point.

In fact, in this case the whole construction is as follows: With centre B (A and B now coinciding), and at the distance B C, describe a circle; then any straight line, drawn from the centre B to the circumference of this circle, is a solution of the problem. This is the case (you will meet with the like again) in which, under special circumstances, the number of solutions to a problem becomes unlimited; sometimes under special circumstances the solution of a problem becomes impossible.

It has been suggested that it might have been better if Euclid had divided this problem into two cases: the first case being that in which the given point was one end of the given line; in this case all the construction required would be to draw a circle with the given point as the centre, and with a radius equal to the given straight line. Then any radius of this circle is a solution of the problem. The second case would be the one solved in Prop. 2, where the given point is *not* one end of the given straight line.

[1] This figure looks complicated, because all the four straight lines are drawn in one figure. It might be simplified by drawing four separate figures, as in the previous exercise.

THE THIRD PROPOSITION DISCUSSED.

The third proposition is one the point of which is often missed. Its dependence on the second is overlooked.

Here, as in the former propositions, take a fresh piece of paper to describe the figure on—a square piece, of the size of an ordinary copy-book, will do; and as there are many lines of construction to be drawn, begin the figure in the middle of the page.

To the question, ever to be repeated, What is given in the third proposition, and what has to be done? the answer is given in the

General Enunciation.

Two straight lines are given, one of which is greater than the other:—

And it is required from the greater, to cut off a part equal to the less.

Particular Enunciation.

On a page different from that on which the figure is to be drawn, write thus:

'1. Let A B and C L be two straight lines, of which A B is the greater.'

Then draw these two straight lines [let C L be about an inch and a half long, and let the nearest extremities of the straight lines be about an inch apart]; and add,

'2. It is required from A B, the greater, to cut off a part equal to C L, the less.'

Construction.

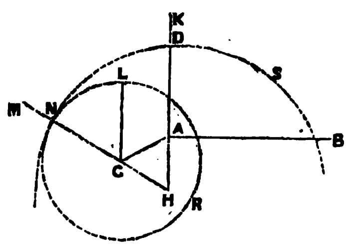
The first step in the construction is:

'I. From the point A, draw a straight line equal to C L (Prop. 2)'

This at once links this, the third proposition, on to the second, for what you are here *told to do* is exactly what you were *taught to do* in the second proposition.

Regard then at present only the *point* A (which is one extremity of the straight line A B) and the straight line C L. And taking A as the given point, and C L as the given straight line, do all the construction of the second proposition over again.

Obs.—In the figure this construction is given in dotted lines; do you do it in pencil, that it may afterwards be effaced.



i. The first step of the construction of the second proposition is, from the point A to C, to draw the straight line A C (Post. 1).

ii. The second step is to describe an equilateral triangle on A C.

To do this we must go back to the first proposition. But in order not to encumber your figure with circles, find the vertex of the equilateral triangle by the intersection of two small arcs according to the method described on p. 30; call the intersection of the arcs H, join H A, and to save time and ensure correctness, *before* removing the ruler, produce H A to any point K. In like manner join H C, and produce it at once to M.

iii. The third step in the construction of the second proposition is this: With the centre C, and at the distance C L, to describe a circle L N R intersecting C M in the point N.

iv. And the last step is, with the centre H, and at the distance H N, to describe the circle N D S intersecting A K in the point D.

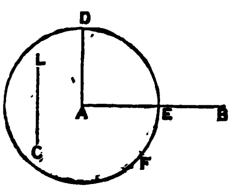
By the above construction A D has been drawn from A equal to C L.

It may be as well to go over to yourself rapidly the

proof given in Prop. 2, that A D is equal to C L. If you are by this time up in the proof, and can follow it with this figure, it will not take a minute. If you are not up in the proof, and are obliged to refer to the demonstration given in the second proposition, it will be good practice to go over it again. And by all means pick out the Windsor man and the London man, and the friend that went by the train. When you have found them out, or think you have, read on and see if you are right.

Ans.—C L and A D are the London man and Windsor man respectively, and C N is the man that went by the train.

And now all the construction indicated by the dotted



lines having been done by you in pencil, ink in the line A D, and when the ink is dry, efface all the pencil construction, and you have remaining on your paper the two given straight lines A B and C L; and a third straight line A D, drawn from one extremity A of the greater straight line A B, equal to C L, the less.

And here begins the new part of the third proposition.

There is only one step of construction in this new part; write it thus:

'2. With the centre A, and at the distance A D, describe the circle D E F, cutting A B in the point E.'

Describe it. You have but to add:—

'Then A E shall be equal to C L.'

Demonstration.

The demonstration is as follows:

'1. A E is equal to A D, because they are both radii of the same circle D E F.

'2. But A D was made equal to C L (Prop. 2).

'3. Wherefore A E and C L are each equal to A D.

'And if they are each equal to the same thing they are equal to one another. (Ax.1.) Therefore A E is equal to C L.

'That is, from A B, the greater of two given straight lines, a part A E has been cut off equal to C L, the less, which was to be done.'

It is hoped that the learner will have no difficulty in detecting that C L and A E are respectively the London man and the Windsor man, and that A D is the man that went by the train.

It only remains now to write out the third proposition, unencumbered with explanatory remarks.

THIRD PROPOSITION WRITTEN OUT.

PROPOSITION III. PROBLEM.

General Enunciation.

Given two straight lines, of which one is greater than the other:—

It is required from the greater to cut off a part equal to the less.

Particular Enunciation. (See fig. p. 40.)

Let A B and C L be two given straight lines, of which A B is the greater:—

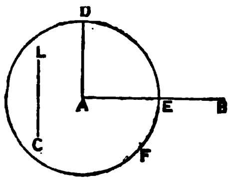
It is required from A B, the greater, to cut off a part equal to C L, the less.

Construction.

1. From the point A draw the straight line A D equal to C L (Prop. 2).
2. From the centre A, and at the distance A D, describe the circle D E F, cutting the straight line A B in the point E. Then shall A E be equal to C L.

Demonstration.

1. A E is equal to A D, because they are both radii of the same circle D E F (Def. 15).



2. But A D was made equal to C L (Prop. 2).

3. Wherefore A E and C L are each of them equal to A D.

4. And things which are equal to the same thing are equal to one another; therefore A E is equal to C L (Ax. 1).

Wherefore from the greater of two given straight lines a part has been cut off equal to the less, which was to be done, or Q. E. F. (*Quod erat faciendum*).

NOTE.—If both the given straight lines are drawn from the same point, the only construction required is the following:

Take that point as centre, and the straight line, which is less than the other, as radius, and describe a circle. This circle will cut off, from the greater of the two straight lines, a part equal to the less (Def. 15).

On the distinction between 'therefore' and 'wherefore.'

'Therefore' is the stronger word, and is used to introduce the sentences in which Euclid clenches the successive steps of the reasoning leading on to the conclusion to be arrived at.

'Wherefore' is the milder word, equivalent to 'thus' or 'so that,' and is thus used in Euclid: when he has shown in *separate sentences* that two lines (or angles) are equal to two others, each to each, and re-states that they are so, in *a single sentence*, he begins it with 'wherefore.' Or if he has proved that two triangles are equal in *every* respect, and has to specify any *particular* respect in which they are equal, he begins with 'wherefore.' Again, when at the close of a proposition he recapitulates what he has done, he introduces the sentence with 'wherefore.'

THE FOURTH PROPOSITION DISCUSSED.

We pass on now to the fourth proposition. This is a great and most important proposition, and has been called the key to Euclid. It is in fact the real beginning of Euclid, the three first propositions being introduced before it only because in the fifth and sixth propositions Euclid has from the greater of two straight lines to cut off a part equal to the less.

The fourth proposition is one of a new kind; unlike the first three, which are *problems*, this is, a *theorem*. In the first three something was to be *done*. In the fourth something has to be *proved*.

General Enunciation.

Let us begin with the never-to-be-omitted question, What is given in this proposition, and what has to be proved?

The reply is as follows:

Two triangles are given, which have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal:—

And it is required to prove that, in this case, the two triangles will be equal in every respect, that is to say—

(1) The bases, or third sides, will be equal.

(2) Each pair of the remaining angles, to which the equal sides are opposite, will be equal.

(3) The triangles will be equal; that is, the quantity of surface enclosed between the two sides and the base of one of the triangles, will be equal to the quantity of surface enclosed by the two sides and the base of the other triangle.

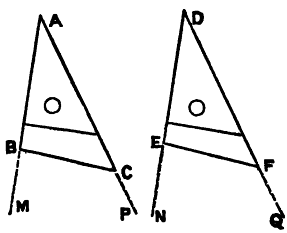
Obs. It may be useful at this point to look back to page 13 and read over again the observations 2 and 3 on Triangles.

Particular Enunciation.

Let A B C and D E F be two triangles which have two sides, B A and A C of the one, equal to two sides, E D and D F of the other, each to each, namely A B to D E, and A C to D F. Also let the angle B A C, contained by the two sides B A, A C of the triangle A B C, be equal to the angle E D F contained by the two sides E D, D F of the triangle D E F:—

Before going on to say what has to be proved, it will be well here to pause and give some directions about drawing these two triangles.

First draw in pencil two straight lines, A M and D N, then take your set square and lay one of its edges along



A M, bringing one of its acute angles to the point A; and using the triangle so placed as a guide and ruler, draw from A a straight line A P; then remove the triangle, and place the same edge of it along the straight line D N, placing on D the angular point which before lay on A, and, in the same manner, from D draw the straight line D Q. Then take your compass and with it mark off from A M and D N equal parts A B and D E. Having done this, open the compass a little more, and mark off from A P and D Q equal parts A C and D F, and join B C and E F. This being done, ink in the sides of triangles A B C and D E F, and effacing the pencil marks denoted in the figure by dotted lines, you have two triangles A B C, D E F, which have two sides of the one equal to two sides of the other, each to each, and have likewise the included angles equal.[1]

[1] It was said, in the observations on the postulates, that your compass and ruler were not to be used for measuring. That only refers to lines and angles which have to be made or proved equal, not to those which are *given* equal; when lines and angles are given equal, you may use any mechanical means for making them so.

And you have to prove:—

First, that the base B C is equal to the base E F.

Secondly, that the other angle A B C is equal to the other angle D E F (these angles being opposite to the equal sides A C and D F); and that the remaining angle A C B is equal to the remaining angle D F E (these angles being opposite to the equal sides A B and D E).

Thirdly, that the triangle A B C is equal to the triangle D E F; that is, that the space or area enclosed by the three straight lines B A, A C, and C B is equal to the area or space enclosed by the three straight lines E D, D F, and F E.

And now, supposing that you have carefully gone through this long enunciation, have drawn the figure, and that you see distinctly what is given and what you have to prove, it is very possible that your first observation will be as follows:—Euclid gives me these two triangles; he tells me that the two sides, or legs, of one of them are respectively equal to the two of the other, and that these legs are equally drawn asunder; and he gives me a long proposition to prove that the lines drawn across, at the bottom of the legs must be equal. Who would think of denying it? It must be so. Anybody in his senses must see that the cross lines at the bottom must be equal.

To this observation the reply is:

What you say is natural, reasonable, and quite true, and (it might be added) shows active intelligence on your part. Yet, nevertheless, if instead of resting satisfied with '*feeling*' it must be so, you will listen to Euclid *patiently*, and first let him tell you the axiom by which he is going to prove the proposition; then follow him as, step by step, he proceeds to prove that the bases, the remaining angles, and the triangles are equal, in accordance with that axiom: —If you take in all the steps of his reasoning, so as to be able to reproduce them correctly:—If moreover you are able to review the whole proposition, and to feel that you have

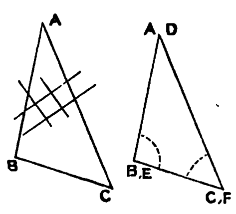
clearly and rigorously proved the truth of the statements he makes in the enunciation:—You will, by this mental process, be lifted up into a higher intellectual level than you have yet reached, or could reach, without this mental effort; and from that higher level you never can come down again all your life long.

Do not think the above to be an exaggerated statement of an enthusiastic teacher of Euclid. There is no mathematician who would not agree that what is here said is no more than *sober truth*.

This is said to encourage you to give attention to the reasoning; to follow what is said with thought and *patience*; and not to allow any statement to pass of which you do not understand the meaning. On the other hand, every step of the reasoning is so simple that there is fear that the simplicity of the steps may be a bar to your taking in the reasoning.

There is nothing abstruse in what you are going to do. Every step of the reasoning is as simple and straightforward as it is rigorous.

Having said this, we will now go back to the proposition. And observe, we have first to prove that the base B C is equal to the base E F.



It is done by the eighth axiom, which asserts, 'That magnitudes which exactly fill the same space are equal' (see the remark on this axiom, page 16). We must therefore show that B C and E F may be made exactly to fill the same space.

With this end in view, imagine the triangle ABC to be taken up and laid upon the triangle D E F, A being placed on D, and the straight line A B being laid along D E, as far as it goes.

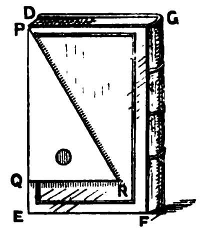
[*Obs.* In the above figure a few strokes are drawn across the triangle A B C. This is done to remind the learner that

he is not to conceive that the triangle A B C is still in its old place, but that it is now lying on D E F.

To impress this on the learner's mind still further, the letters A, B, and C have been placed at the angular points of the triangle D E F; A lying on D, it having been placed there, and B and C lying on E and F, where it will presently be *proved* that they will come.]

It is most important that the learner should, at this point of the discussion of the fourth proposition, see clearly that any figure may be placed on any other figure, as Euclid directs that the triangle A B C shall be placed on the triangle D E F, provided only that they be *rectilineal.*

To illustrate this, let the learner here take up a book D E F G, and his set square P Q R. The book and the triangle are quite unlike in form, yet one of the angular points P of the triangle can be put on the corner of the book, and the triangle can be laid on the book so that a side of the triangle P Q shall lie along the edge of the book D E, *as far as it goes.* This is all that Euclid tells you to do.



It is not uncommon for learners to begin this proposition in some such way as the following: Put A on D, and B on E, and C on F. But this is altogether wrong, for this you could not do with *any* two triangles which you might chance to take up; and what Euclid tells us to do might be done with such chance triangles, or indeed with a triangle and a book, as we have seen.

Returning now to the two triangles A B C and D E F: A is placed on D, and A B is made to lie along D E, as far as it goes. But how far will it go? where will the end of it, B, come? where will it naturally and of necessity fall? and why? That is now the question.

Do you hesitate to answer? Well, then, as Euclid's argument is simply an appeal to our common sense, to our

every-day experience, we will have recourse to a very homely illustration of his reasoning.

Here are a couple of walking-sticks; one of them is two inches longer than the other. Now, if the handles are brought together, and one of the sticks is laid along the other, will the points come together? You feel they will not, and the reason why, you will give readily, 'One of them is longer than the other.' That is quite right.

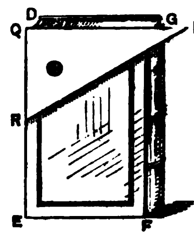
But here are two other walking-sticks; these are of equal length. If we put the handles of *these* together, and lay one of the sticks alongside the other, will *their* points come together? You answer, 'Yes, of course they will.' Why? 'Because the sticks are of equal length.' Exactly so; that is all right. Now turn once again to the triangles. Look upon A B and D E as representing the two walking-sticks; consider the extremities A and D as the handles. Now, A has been placed on D, and A B has been laid along D E, as far as it goes. How far then will it go? where will its extremity B come? will it come on E? 'It will,' you reply, 'if A B is equal to D E.' But if you look at the enunciation you will see that these lines were given equal; so that you *can* say, B will come on E, *because A B was given equal to D E.*

Are you disposed to say, 'Is that all?' It is all. Possibly you may feel disposed to add, 'There is nothing in it.' There is a great deal in it. There is a bit of reasoning perfectly simple, perfectly conclusive. All the succeeding steps are equally so.

We will proceed. Take up again your set square P Q R and the book D E F G. It is, you may observe (see last figure), one of the acute angles of the triangle which has been placed on the corner of the book; and you have one side of the triangle P Q lying along the side edge of the book, as far as they go together. Will the other side of the triangle P R lie along the top edge of the book? Clearly it will not.

But if the triangle is turned round, and the right

angle of the triangle is placed on the corner of the book, and if, as before, one side Q R of the triangle is made to lie along the side edge D E of the book, does the other side Q P of the triangle lie along the top edge of the book, as far as they go together? Yes, now it does.

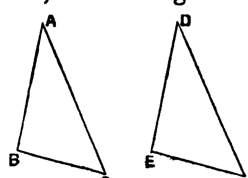


And why is it that in this last case the second sides come together, while in the former they did not? Simply because in the last case the included angles are equal, both being right angles, but in the former they were not.

Turn once again to the two triangles A B C and D E F (see below). We placed A on D, and A B along D E; will A C lie along D F? Certainly, because the included angles B A C and E D F are equal. You remember these angles were given equal, and we took pains, in drawing the triangle, to make them so.

We will pause here to introduce a story. There is often a great deal of teaching to be got from a story.

A boy in going through the fourth proposition said thus: 'Because A B is equal to D E, and the angle B A C is equal to E D F, therefore the line A C lies on D F.' 'No, no,' said the teacher. 'What ought he to say?' The next boy answered, 'Because A B *lies on* D E.' A little fellow sitting



beside the speaker (he is a post-captain in the navy now) here looked up eagerly into the teacher's face, and said, 'Please, sir, will you tell me why it is not as good to say it one way as the other?' 'Do you wish to know? I will tell you willingly,' was the reply. 'Look at those two triangles on the black board, A B C and D E F. You remember A B and A C are respectively equal to D E and D F, and the included angle B A C is equal to the included angle E D F.' He answered, 'Yes.'

'Now,' said the teacher, 'if I were to take a saw, and cut down through the board between the triangles, and were to send one half of the board, namely, that half on which the triangle A B C is drawn, to Oxford, and the other half, that on which D E F is drawn, to Cambridge, the lines and angles would still be equal, would they not? Of course they would. But if one of the triangles were at Oxford and the other at Cambridge, could the line A C *lie on* the line on D F? Certainly not.

'But if the triangles are neither at Oxford nor at Cambridge, but there on the board before us, and I conceive the triangle A B C to be laid on D E F, laying A on D, and A B along D E, then indeed A C will lie along D F, because the included angles are equal.'

A flush suffused the boy's countenance. 'Thank you, sir,' said he. 'I see it perfectly now; I never saw it before.'
—*Narrative Essay on a Liberal Education*, p. 62.

To resume the demonstration. We have now shown that A C will lie on D F, as far as it goes, and given the reason why. The last question is, Will the point C come naturally and of necessity on F? Yes, it will—and why? Because the straight lines A C and D F were given equal.

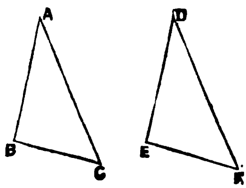
[It would be paying you a poor compliment to take up the walking-sticks again to illustrate this.]

We will here review what we have done. Supposing the triangle A B C laid on the triangle D E F, as Euclid tells us to lay it; that is as *any triangle* whatever might be laid on *any other*, we have shown, in the case of these *two particular triangles*, that—in consequence of their having two sides of the one equal to two sides of the other, and the included angles being equal—*B will come on E, and C on F.*

A little reflection will tell that if *any one* of these three given conditions were wanting, we could not prove that B would come on E and C on F. But all three of them being given, we *can* prove that B *must* come on E, and C on F.

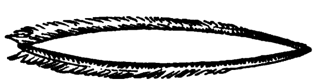
We now go forward:

Look at, and for the present confine your attention wholly to, the two straight lines B C and E F (do not think about the triangles). Observe that B is the beginning and C is the end of one of these straight lines, and that E is the beginning and F the end of the other of them; and it has been shown that the beginning B lies on the beginning E, and that the end C lies on the end F.



Now these two straight lines being *together* at their beginning and at their end, is it possible for them to separate at all between the beginning and the end? 'that is now the question.'

Here are a couple of pens; if the feather ends are brought together and the quill ends are brought together, they *do* separate between the beginning and the end, and by thus separating they enclose or shut up a space; but that is because the pens are not straight. Two *straight* lines cannot enclose a space (Ax. 10); it takes at least three straight lines to enclose or shut up a space. Therefore, the two straight lines B C and E F, being together at their beginning and together at their end, must keep together from beginning to end.



What is the end of all this? Why this: that the two straight lines B C and E F are together at their beginning, and are together at their end, and do not separate between the beginning and the end: and therefore *they fill the same space.*

In fact, if you attempt to draw the point of your pen along the line joining B and C, and again along the line joining E and F, you find that you have in both cases drawn the point of your pen through *the same space.* But magni-

tudes which fill the same space are equal. Therefore the base B C is equal to the base E F (Ax. 8).[1]

We have thus proved, in accordance with the eighth axiom, that the base B C is equal to the base E F. It will be a slight task compared with the foregoing to prove that the remaining angles, to which the equal sides are opposite, are equal, and that the triangles are equal.

These are proved equal by the same axiom; we must therefore see if they fill the same space. First let us consider the angles A B C and D E F. Confine your attention wholly to these angles: they are contained by the straight lines A B, B C, and D E, E F respectively.

Now ask yourself the question, where is A B lying? The answer is, it is lying along D E, for *we put it there.* And where is B C now lying? It is lying along E F; *we have just proved that it must come there.* Whence it must follow that the angles A B C and D E F fill or occupy the *same* space. For if now you make (see fig. p. 46) the arc, or curved rim, described page 6, to indicate the angle contained by the straight lines A B, B C; that is, the angle A B C; and again, if you make a similar curved rim to indicate the angle contained by the straight lines D E, E F, that is, the angle D E F, you find that you draw the curved rims in both cases through the *same* space; that is, the angles A B C and D E F fill the *same* space, and therefore, in accordance with the eighth axiom, they are equal.

Again the angles A C B and D F E are proved to be equal in precisely the same manner. A C is now lying on D F [not because we put it there, as in the case of A B and D E, but] because, having put A B on D E, we proved that A C must lie along D F, and C B, as said above, has been proved to lie along F E, so that the two rims or arcs, if drawn to indicate the angles A C B and D F E, would both be drawn through the same space, thereby showing that the

[1] If we show that two lines begin together, end together, and keep together from beginning to end, this is quite enough to prove that they fill the same space; for lines have no thickness; they can only fill or occupy space of one dimension—that is, length.

two angles A C B and D F E both fill the *same* space, and therefore again by the eighth axiom they are equal.

Once more. In order to prove that the triangles are equal, we follow the same line of reasoning, thus :—The straight line A B was placed on D E, and hence we proved that A C must lie along D F, also that B C must lie along E F; hence evidently if you put your finger first on the space enclosed or shut up within the three straight lines A B, B C, and C A, and again on the space shut up or enclosed within the three straight lines D E, E F, and F D, you find you are putting your finger, in both cases, in the same space; that is, the triangles A B C and D E F both fill the same space,[1] and therefore they are equal (Ax. 8).

It is all over now. You have thoroughly and rigorously proved, in accordance with the eighth axiom, that, if two triangles have two sides of the one equal to two sides of the other, and have likewise the angles contained by those sides equal, they are equal in every respect—that is, in respect of sides, in respect of angles, and in respect of area.

It will be useful to notice the three following remarks:—

1. In order to prove that two lines fill the same space, it is necessary to show that they begin together, that they end together, and that they keep together from beginning to end.

2. In order to prove that two angles fill the same space, it is necessary to show that the two lines containing one angle lie, as far as they go, on the two lines which contain the other angle.

3. In order to prove that two areas fill the same space, it is necessary to show that the three boundary lines which enclose one of the areas, lie along the three boundary lines which enclose the other area.

It remains to write out the proposition for your guidance in the form in which it might be written in an examination.

[1] It is only necessary to show that the boundaries (see Def. 13) of the triangles lie together, in order to prove that the triangles fill the same space, for triangles are surfaces which have only length and breadth and no thickness.

THE FOURTH PROPOSITION WRITTEN OUT.

PROPOSITION IV. THEOREM.

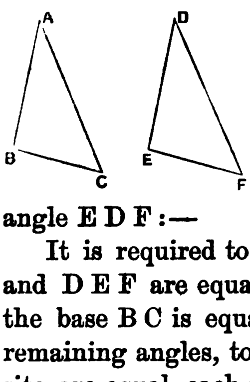
General Enunciation.

It is given that two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal:—

It is required to prove that the triangles are equal in every respect; that is, that their bases are equal, that their remaining angles are equal, each to each, viz., those to which the equal sides are opposite, and that the triangles are equal.

Particular Enunciation.

Let A B C and D E F be two triangles which have the two sides B A, A C, equal to the two sides E D, D F, each to each, viz., B A to E D, and A C to D F, and have likewise the included angle B A C equal to the included angle E D F:—



It is required to prove that the triangles A B C and D E F are equal in every respect; that is, that the base B C is equal to the base E F, and that the remaining angles, to which the equal sides are opposite, are equal, each to each, viz., A B C to D E F, and A C B to D F E; also that the triangles A B C and D E F are equal.

Construction.

Apply the triangle A B C to the triangle D E F, so that the point A may be on the point D, and that the straight line A B may lie along the straight line D E.

Demonstration.

1. Then the point B will come on the point E, because the straight lines A B, D E are equal (hyp.).

2. And A B lying along D E, A C will lie along D F, because the included angles B A C and E D F are equal (hyp.).

3. And A C lying along D F, the point C will come on the point F, because the straight lines A C and D F are equal (hyp.).

4. And it has been proved that B will come on E.

5. And B lying on E, and C on F, the straight line B C must lie along the straight line E F from beginning to end, for if they separated, they would enclose a space, which is impossible (Ax. 10).

6. Wherefore the two straight lines B C, E F—beginning together, ending together, and keeping together from beginning to end—fill the same space, and are therefore equal (Ax. 8).

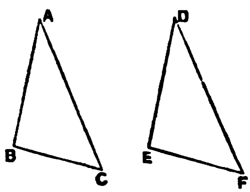
7. Also A B, B C lying along D E, E F, the angles A B C and D E F fill the same space, and are therefore equal (Ax. 8).

8. Likewise A C, C B lying along D F, F E, the angles A C B and D F E fill the same space, and therefore are equal (Ax. 8).

9. And the three boundary lines A B, B C, C A lying along D E, E F, F D, the triangles A B C and D E F fill the same space, and are therefore equal (Ax. 8).

Wherefore if two triangles have two sides of the one equal to two sides of the other, and have likewise the angles contained by those sides equal, they are equal in every respect, which was to be shown—or *quod erat demonstrandum*, or Q.E.D.

N.B. The words of the enunciation 'to which the equal sides are opposite' must not be overlooked. They enable us to say which of the two remaining angles of each triangle form the pair of equal angles. Here A C is given equal to D F. Now the angle opposite to A C is A B C, and the angle opposite to D F is D E F. Hence the angles A B C and D E F, which are opposite the equal sides A C and D F, form *one* pair of the remaining angles which are equal to each other. Similarly the angles A C B and D F E, opposite to the equal sides A B and D E, form the *other* pair of the remaining angles which are equal to each other.



The readiest way then to find which are the remaining angles that are equal to each other, is to look for the equal sides; then the angles which are opposite to the equal sides are the angles which are equal to each other.

*** It is very important to exclude all thought of thickness when discussing the fourth proposition. Triangles are surfaces, and have no thickness. It is the surfaces coming in contact which fill the same space.

To make this clear to young minds a correspondent suggests the following method of describing the superposition of triangle A B C on D E F:

Instead of saying 'apply the triangle A B C to D E F' he proposes to say 'wipe out the triangle D E F, but remember exactly where it *was*, and where the angular points *were*. Then place the point A where D *was*, and let A B run exactly where D E *ran*. Then B must fall into E's place because,' &c., and so on.

Appended Note on the Fourth Proposition of Euclid, addressed to Teachers.

As the writer has departed a good deal from the usual phraseology in the fourth proposition, especially in avoiding the use of the word 'coincide,' it is right that he should state his reasons.

The fourth proposition will have done its great good to the learner's mind, when he has fully taken in the meaning of the eighth axiom, and felt that, in every respect in which the triangles have to be proved equal, the reasoning has shown that the axiom is fulfilled.

In order to help a learner to do this, he thinks that every obstacle to the right understanding of the axiom, and of the reasoning by which its application is shown, should be put aside.

Now the use of the word 'coincide' in the commonly received phraseology of the fourth proposition is (according to the writer's experience) an obstacle to the learner's taking in the meaning of the axiom, and the reasoning by which it is applied, which in *kindness* to him had better be removed.

In the course of the demonstration the word coincide is used in three different senses, besides its own rigorous, technical sense, as given in the eighth axiom.

First, it is used for coincide in position. Secondly, it is used for coincide in direction. Thirdly, it is used to convey the idea of not separating.

It is not till the fourth time of using the word that it is used in Euclid's full, technical sense of exactly filling—occupying wholly—the same space.

The result of this is, that the minds of learners become so confused that they are as ready to say, at the close of the proof, 'are equal and coincide,' as to say 'coincide and are

equal;' indeed they almost invariably, when proving that the remaining angles, and that the triangles are equal, leave out the word 'coincide' altogether;—by these mistakes showing that they have wholly failed to take in the meaning of the demonstration.

Teachers will readily bear witness how often the bewildered pupil (when corrected) replies, 'But to coincide means to be equal.' Now, as coincide means no such thing, the writer has thought it better in this treatise to remove this difficulty out of the learner's way by using for the word 'coincide' what it really does mean.

It may be objected that by using, instead of the word coincide, the synonymous expression, 'exactly fill the same space,' he brings into prominence what is gently disguised by the word coincide; viz., that certain magnitudes (lines and angles) are said to fill a space, when from their very nature they can fill no space.

The writer holds this to be as nothing compared with a right appreciation of the axiom and its application.

It may be true that lines, having length only, and no breadth, can fill no space; yet if it is proved that they begin together, that they end together, and continue together from beginning to end, then it is shown in an intelligible and very real way that the two lines do exactly fill, or wholly occupy, the same space *as far as their nature enables them to do so.*

Again, if it is shown that the two straight lines containing one angle, lie (as far as they go) exactly along the two straight lines which contain another angle, then we show in a very real and intelligible way that the two angles fill the same space *as far as angular magnitude can do so.*

Indeed, although the lines containing the angles to be proved equal do actually, in the fourth proposition, coincide according to Euclid's full, technical meaning of the word, yet the writer thinks it positively desirable, with the view of keeping the learner's ideas clear on this point, to put out of view the technical, full coincidence of the containing

lines. It is their coincidence in direction (or according to our familiar phrase 'as far as they go') that is alone needed, in order to show that the angles contained by them exactly fill the same space, in the only way in which the magnitudes under consideration (angular magnitudes) can do so.

So when we come to the coincidence and consequent equality of the triangles, that is of the enclosed areas, the pointed, prominent thought that should be brought before the pupil's mind is, *not* that the straight lines enclosing the space do *per se* fulfil Euclid's full definition of coincidence, but that, *considered as the boundary lines* of the enclosed spaces to be proved equal, they lie one upon and along the other; by this means showing that the enclosed areas (and it is on the enclosed areas that the pupil's thoughts should now be concentrated) fill the same space.

The question whether pupils will get more readily a true insight into the meaning of the fourth proposition by using or not using the word 'coincide' must be left to the experience of teachers, '*solvitur docendo.*'

EXERCISES ON THE FOURTH PROPOSITION.

Supposing that you have now mastered the fourth proposition, you have next to be told that as you go on through Euclid you will very often have to employ this proposition to prove that triangles occurring in subsequent propositions are equal in every respect.

Henceforth whenever you can detect that two triangles occurring in any proposition have two sides of the one equal to two sides of the other, and have likewise the angles contained by those sides equal, you can say at once: '*Therefore by the fourth proposition the two triangles are equal in every respect;*' for that is what the fourth proposition has proved.

Now the fifth and sixth propositions are nothing but exercises on the fourth proposition. You have to prove what is stated in the enunciations of those propositions by showing that certain triangles occurring in them are equal in every respect by the fourth proposition. But they are hard exercises. Many much easier exercises occur in subsequent propositions. These subsequent propositions, however, could not be introduced before the fifth, because they depend on the fifth, as you found the third proposition to depend on the second.

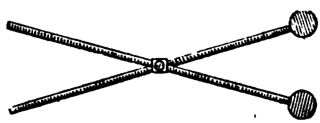
And although, on this account, in a regular treatise, these subsequent propositions could not be introduced before the fifth, there will be no harm in bringing in here (in this introduction to Euclid) such propositions or fragments of them, if you know that the object of giving them to you now is to help you, and to lead you on to harder exercises, and that they will afterwards come before you, in due course, in their proper places in Euclid. It will, however, be necessary, in this case, that, for the present, you

take for granted a few things that you will hereafter prove in Euclid. Such as these:

1. That you know how to draw a straight line cutting an angle into two equal angles; or, at least, that it can be done.

2. That a straight line can be drawn perpendicular to a given straight line.

3. That if two straight lines cross each other like the legs of a scissors, the angle or opening between the blades is always equal to that between the handles.

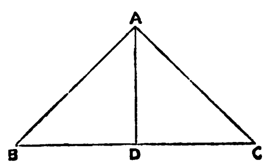


4. That a square may be described on a given straight line.

With these preliminary remarks we will now put before the pupil a graduated series of exercises on the fourth proposition.

EXERCISE I.

Let A B C be a triangle in which the side A B is equal to A C, and taking for granted (for the present) that we can and have drawn the straight line A D so as to divide the whole angle B A C into two equal angles, viz., B A D and D A C [remember, it is the *angle*, not the *triangle*, that you are to suppose divided into two equal parts]:—



It is required to prove that B D is equal to D C.

In this and all the subsequent exercises the following rules are to be attended to:

1. Observe the two triangles you are about to compare, and find out which of the three sides is to be the base of each triangle. In this exercise the two triangles to be compared are A B D and A C D; and there is no doubt here which are the bases, for we are told to prove that B D is

equal to D C; and the sides to be *proved equal* are always to be taken as the bases.

2. Having found which is the base, then the two sides and the angle contained by the two sides are immediately known.[1] Refer now to the enunciation, in order to find if any statement is there made which makes you know that one of the sides of one of the triangles is equal to one of the other. On doing so, you find that the side A B is given equal to A C. It is so stated in the enunciation.

Write therefore as follows:

'1. A B is equal to A C, for it was so given or stated in the enunciation; or more briefly, "by the hypothesis," or simply (hyp.).'

Then do the same for the second side. In this exercise, the second side A D is a side of each of the triangles; in other words, is common to both.

Therefore write next:

'2. A D is common to both triangles.'

Next it is desirable, especially at first, in order to fix in your mind what the two sides are, to repeat Steps 1 and 2, only now taking *together* the two sides of each triangle.

Write therefore thus:

'3. Wherefore the two sides, B A, A D of the triangle B A D are equal to the two sides C A, A D of the triangle C A D.'

Your naming the two sides of each triangle thus, makes you see what the angle contained by the two sides—in other words, the included angle—is; for when you have said that the two sides are B A, A D, you see that the included angle is B A D; and so in the other triangle when you have said that the two sides are C A, A D, you see that the included angle is C A D. Look then back to the enunciation, and see if you are there told that the included angle B A D is equal to C A D. You there find that it is given or supposed that these angles are equal, and you therefore write thus:

[1] See Observation 2, on triangles, p. 13.

'4. And the included angle B A D is equal to the included angle C A D, because it was given that the whole angle B A C was divided into two equal parts, or bisected, by the straight line A D.'

Having thus shown that in the two triangles A B D and A C D two sides of the one are equal to two sides of the other, and that the angles contained by those sides are equal, you conclude the exercise thus:

'5. Therefore by the fourth proposition the two triangles A B D and A C D are equal in *every* respect; wherefore the base B D is equal to the base C D; which was to be proved.'

If in this exercise, instead of being told to prove that the base B D was equal to D C, you were told to prove that A D was at right angles to B C; the exercise would be done exactly as explained above, as far as the words, 'Therefore, by the fourth proposition, the two triangles are equal in every respect.' But instead of closing with the words, 'Wherefore the base B D is equal to the base C D,' you would have to close the exercise thus:—'Wherefore the other angles, to which the equal sides are opposite, are equal; that is, the angle A D B is equal to the angle A D C;' you would have to add, 'and these are adjacent angles; but when one straight line falling on another straight line makes the adjacent angles equal to each other, each of them is a right angle (Def. 10). Wherefore A D is at right angles to B C, which was to be proved.'

Observe that although in this form of the exercise you are not told which are the bases, there is no difficulty in finding them out; for the bases are always the sides opposite to the angles *given* equal. Here the angles given equal are B A D and C A D, therefore B D and D C, the sides opposite to them, are the bases.

The above exercise has been so interspersed with explanatory remarks that it may be well to write it out, as it might be shown up for an exercise.

THE FIRST EXERCISE WRITTEN OUT.

Enunciation.

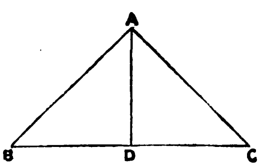
It is given that A B C is an isosceles triangle having the sides BA, AC equal, and that the angle B A C is bisected by the line A D :—

It is required to prove that the base B D is equal to the base C D.

Construction (none).

Demonstration.

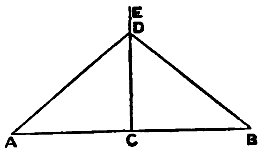
1. A B is equal to A C (hyp.).
2. A D is common to both triangles.
3. Wherefore the two sides B A, A D of the triangle B A D are equal to the two sides C A, A D of the triangle C A D.



4. And the included angle B A D is equal to the included angle C A D (hyp.).
5. Therefore by the fourth proposition the two triangles B A D and C A D are equal in *every* respect; wherefore the base B D is equal to the base C D, which was to be proved.

In the following exercises a few hints will be given; and it is advised that, with the help of those hints, the learner should try to write out the five steps of the demonstration. The five steps are written out in each case, fully, in this book, in order that the learner may see if he is right. If in any case he finds the exercise too difficult to be done without help, he can refer to the book for assistance; still let him not lose courage, but try to do the next exercise without help.

EXERCISE II.



Let A B be a straight line, of which C is the middle point. Let it be granted that C E is drawn at right angles to A B from the point C. Let D be any point in E C, and let A D and B D be joined:—

It is required to prove that A D is equal to B D.

Hints for the solution of Exercise II.

1. Here the two triangles to be compared are clearly A C D and B C D, and A D, D B are the bases of the triangles, for these are the sides to be proved equal.

2. These being the bases, the two sides of the triangle A C D are A C, C D, and the two sides of the triangle B C D are B C, C D, and the angles contained by these sides are A C D and B C D.

3. Now of these sides, A C and C B are given equal (see ennunciation); also C D is a side of both triangles; and the included angle A C D is equal to the included angle B C D because they are right angles (Ax. 11).

With these hints let the learner try to write out the five steps of the demonstration without, for the present, reading further. When he has written them out, he can read on and see if he is right.

DEMONSTRATION OF EX. II. WRITTEN OUT.

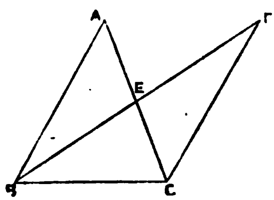
(See above Figure.)

1. A C is equal to B C (hyp.).
2. C D is common to both triangles.
3. Wherefore the two sides A C, C D of the triangle A C D are equal to the two sides B C, C D of the triangle B C D, each to each.

4. And the included angle A C D is equal to the included angle B C D, because C E is at right angles to A B (hyp.).

5. Therefore by the fourth proposition the two triangles A C D and B C D are equal in *every* respect; wherefore the base A D is equal to the base B D, which was to be proved

EXERCISE III.



Let A B C be a triangle. Let it be granted that E is the middle point of A C, join B E, and produce B E to F, making E F equal to B E (this can be done by Prop. 3), and join F C:—

It is required to prove that the angle B A E is equal to the angle E C F, and that the angles A B E and E F C are also equal.

In the demonstration of this exercise you may take for granted that when a pair of scissors are opened, the angle between the blades is equal to the angle between the handles; in other words, that when two straight lines as A C and B F cross each other or intersect, the vertical or opposite angles A E B and C E F are equal. This will be proved in the fifteenth proposition.

Hints for the Solution of Exercise III.

We have first to find out which are the two triangles to be compared. Clearly they are the triangles A E B and C E F, for the angles to be proved equal are angles of these triangles.

Secondly, we have to find out which of the boundary lines in these triangles are to be taken as the bases.

It may be well here to enumerate the different rules which may be given for discovering the bases, and to show that they all point to the same lines.

Rule I.—We know that in the fourth proposition the angles to be *proved* equal are the angles at the base; that is,

the angles of each of which the base is one side; but here the two angles E A B and E B A (of each of which AB is a side) have to be proved equal to the two angles E C F and E F C (of each of which C F is a side). This rule, therefore, points out A B and C F as the bases of the triangles to be compared.

Rule II.—The bases are the sides opposite to the angles which are given equal. In this exercise the angles given equal are the vertical and opposite angles A E B and C E F. This rule then also points to A B and C F, which are the sides opposite these angles, as the bases.

Rule III.—Of the three sides of one of the two triangles that which is *not* given equal to a side of the other, is the base. In this exercise A E is given equal to E C, and E F is made equal to B E. This rule also points to A B and C F (the sides not given equal) as the bases.

And, knowing which of the three boundary lines are to be taken as the *bases*, we know which are the two *sides* in each triangle, and which are the included angles.

With the above hints, let the learner try to write out the five steps of the demonstration, without, as yet, reading further.

DEMONSTRATION OF EX. III. WRITTEN OUT.

(See preceding Figure.)

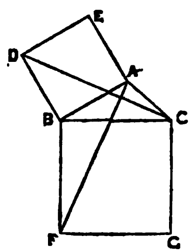
1. The side A E is equal to the side E C (hyp.).
2. The side B E is equal to the side E F (construction). [They were made so by Prop. 3.]
3. Wherefore the two sides A E, E B of the triangle A E B are equal to the two sides C E, E F of the triangle C E F, each to each.
4. And the included angle A E B is equal to the included angle C E F, because they are vertical and opposite angles [which in this demonstration we are allowed to consider equal].

5. Therefore by the fourth proposition the two triangles A E B and C E F are equal in every respect. Wherefore the other angles *to which the equal sides are opposite* are equal; that is, the angle B A E is equal to the angle F C E, and the angle A B E to the angle C F E; which was to be proved.

Exercise IV.

In this exercise it is allowed, or assumed, that a square may be described on a given straight line. The way to do this will be shown in the forty-sixth proposition.

Enunciation.



A B C is any triangle. It is assumed that on the two sides of it, A B and B C, squares are described; viz., A E D B and B F G C.

Also D C and A F are joined:—

It is required to prove that D C and F A are equal.

Hints for the solution of Exercise IV.

D C and A F, being the straight lines to be proved equal, are therefore the bases of the triangles to be compared. Now the triangles of which D C and A F are bases are D B C and A B F. These, then, are the triangles to be compared. And as we know that D C and A F are the bases, we know at once that D B, B C are the two sides of the triangle D B C, and that A B, B F are the two sides of the triangle A B F.

We must now examine if these sides, and the angles included by them, are known to be equal.

1. Is D B equal to A B? Yes, they are two sides of a square (see Def. 30).

2. Is B C equal to B F? Yes, for the same reason.

3. Is the angle D B C equal to the angle A B F? This is not evident at first sight, but a little reflection will show that it is. For the angle D B A is equal to F B C (see Def. 30 and Ax. 11); and if to each of these equal angles the angle A B C be added, we know (by Ax. 2) that the whole angle D B C is equal to the whole angle A B F.

With these hints, let the learner now try to write out the five steps of the demonstration without reading further at present.

DEMONSTRATION OF EX. IV. WRITTEN OUT.

(See preceding Figure.)

1. The side A B is equal to the side D B, because they are two sides of the same square.

2. The side B F is equal to the side B C, because they are two sides of the same square.

3. Wherefore the two sides A B, B F of the triangle A B F are equal to the two sides D B, B C of the triangle D B C.

4. Also the included angles A B F and D B C may be proved equal, thus: the angle F B C is equal to the angle D B A because they are angles of squares, and the angles of squares are right angles (Def. 30), and all right angles are equal (Ax. 11).

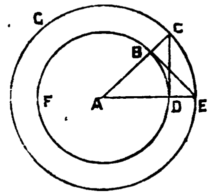
And if to each of these equal angles the angle A B C be added, then the whole angle A B F is equal to the whole angle D B C (Ax. 2), and these are the included angles.

5. Therefore by the fourth proposition the two triangles A B F and D B C are equal in every respect. Wherefore the base A F is equal to the base D C; which was to be proved.

EXERCISE V.

Describe two circles B D F and C E G from the same

centre A, but with different radii. From the common centre of the two circles draw two radii (as in the figure) A C, A E, to the outer circumference, cutting the inner circumference in B and D; join B E, D C:—



It is required to prove that B E is equal to D C; also that the angle A B E is equal to A D C, and that the angle A C D is equal to A E B.

Hints for the solution of Exercise V.

The two triangles to be compared are A B E and A D C, and the bases of the triangles are B E, D C, these being the straight lines to be proved equal.

Hence it follows that D A, A C are the two sides in the triangle D A C, and that B A, A E are those in the triangle B A E; also that the included angle, in each triangle, is the angle at A, which is common to both triangles.

Now, without further hints, try, and for a particular reason that you will know by-and-by, *try hard* to write down the five steps of the demonstration before reading further.

DEMONSTRATION OF EXERCISE V. WRITTEN OUT.

(See preceding Figure.)

1. A B is equal to A D, because they are both radii of the same circle B D F.

2. A C is equal to A E, because they are both radii of the same circle C E G.

3. Wherefore the two sides B A, A E of the triangle B A E are equal to the two sides D A, A C of the triangle D A C.

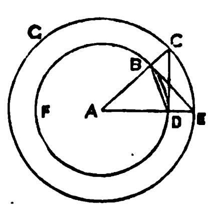
4. And the included angle B A E is equal to the included D A C, both being one and the same angle [or, in other words, the included angle at A is common to both triangles].

5. Therefore by the fourth proposition the two triangles A B E and A D C are equal in every respect; wherefore the base B E is equal to the base D C, and the remaining angles, to which the equal sides are opposite, are equal; viz., the angle A D C to the angle A B E, and the angle A C D to the angle A E B; which was to be proved.

EXERCISE VI.

The sixth exercise is to be taken in connection with the fifth.

The only additional construction is, join B D; and taking as demonstrated all that was proved in the fifth exercise:—



It is required to prove that the angle C B D is equal to the angle E D B, and that the angle C D B is equal to the angle E B D.

Hints for the solution of Exercise VI.

The two triangles to be compared in this exercise are, clearly, the triangles C B D and E D B. The next question is, What are the bases of these triangles? They may be discovered by the following consideration (among others):

In Exercise V. it was demonstrated that the angle A C D is equal to the angle A E B, which is the same as saying that the angle B C D is equal to the angle D E B; that is, in the two triangles to be compared, the angle B C D has been demonstrated to be equal to D E B. And since in any two triangles to be compared by the fourth proposition the angles which are given, or are known to be, equal are opposite to the bases, it follows that B D is the base of each of the triangles C B D and E D B.

And B D being the base of the triangle C B D, its two sides are B C, C D; also B D being the base of the triangle E D B, its two sides are D E, E B.

Now, before we can say that, by the fourth proposition, the triangles C B D and E D B are equal in every respect, we must show that the two sides B C, C D of the one are equal to the two sides D E, E B of the other, each to each. Can we do so?

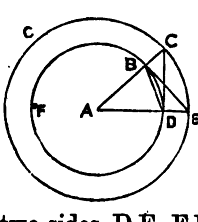
First. Is B E equal to C D? It is. These lines were proved equal in the fifth exercise; they are the bases of two triangles A D C, A B E, proved equal in every respect by the fourth proposition.

Secondly. Is B C equal to D E? It is; for the whole A C is equal to the whole A E (Def. 15), and A B a part of A C is equal to A D a part of A E (Def. 15); therefore the remainder B C is equal to the remainder D E (Ax. 3).

With the above hints to guide you, try now to write out the five steps of the demonstration before reading further.

DEMONSTRATION OF EXERCISE VI. WRITTEN OUT.

1. The side D C has been demonstrated to be equal to B E (Exercise V.).



2. The side B C is equal to D E, for the whole A C is equal to the whole A E, of which the parts A B and A D are equal; therefore the remainder B C is equal to the remainder D E (Ax. 3).

3. Wherefore the two sides B C, C D of the triangle B C D are equal to the two sides D E, E B of the triangle D E B.

4. And the included angle B C D has been demonstrated to be equal to the included angle D E B (Exercise V.).

5. Therefore by the fourth proposition the two triangles C B D and E D B are equal in every respect; wherefore the remaining angles, to which the equal sides are opposite, are equal; viz., the angle C B D to E D B, and the angle B D C to the angle D B E; which was to be proved.

DEDUCTION FROM EXERCISES V. AND VI.

(See preceding Figure.)

The following deduction is not an exercise on the fourth proposition. It is introduced here for a special reason, which you will know afterwards.

Enunciation.

It is required to deduce—from the fifth and sixth exercises—that the angle A B D is equal to the angle A D B.

Hints for the solution of the deduction from Ex. V. and VI.

The hints for this deduction are given in the form of five suggestive questions. The answers are put in a footnote; let the learner try to make out the answers himself first, before he looks at the note below.

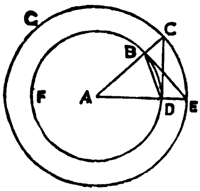
1. In the fifth exercise, what angle was proved equal to A B E?
2. And in the sixth exercise, what angle was proved equal to D B E?
3. Now, if from the angle A B E the angle D B E be taken, what angle remains?
4. And if from the angle A D C the angle B D C be taken, what angle remains?
5. And if equals be taken from equals, what do we know about the remainders? [1]

The demonstration follows in due form on the next page. Try to write it out before reading further.

[1] *Ans.* 1. The angle A D C. *Ans.* 2. The angle B D C. *Ans.* 3. The angle A B D. *Ans.* 4. The angle A D B. *Ans.* 5. That they are equal.

DEMONSTRATION OF DEDUCTION FROM EXERCISES V. AND VI. WRITTEN OUT.

1. The whole angle A B E has been demonstrated to be equal to A D C (Exercise V.).



2. And the angles D B E and B D C, parts of these, have also been demonstrated to be equal (Exercise VI.).

3. Therefore the remaining angle A B D is equal to the remaining angle A D B (Ax. 3). Q.E.D.

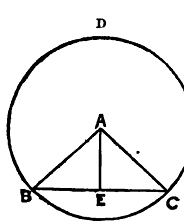
Last Exercise.

Before bringing these exercises to a close, one more is introduced to test the learner's knowledge of the fourth proposition, and his power rightly to apply it. The more so, as there is a story connected with it.

It is only fair here to warn the student that this last exercise is one which cannot be solved by the Fourth Proposition. It is here introduced in order that the pupil may exercise himself in finding out the mistake of those who tried to solve it by the Fourth Proposition.

The exercise is as follows:

Enunciation.



Let B C D be a circle of which A is the centre, and let B C be a straight line terminated by the circumference, and not passing through the centre, and let it be granted that A E is perpendicular to B C:—

It is required to prove that B C is bisected in E.

Construction.

Join A B and A C.

Now, this exercise is really the third proposition of the third book of Euclid. And the story connected with it is this, that the question was set in an examination for a very distinguished honour at Cambridge. The candidates were selected men, yet all of them but two attempted to prove it by an incorrect application of the fourth proposition.

The attempted solution of those candidates who did the proposition wrong is here put down; and the learner is strongly advised to try, without help, to find out what was their mistake, and why it is a mistake.

The *incorrect* solution is as follows:

1. The side A B is equal to the side A C because they are radii of the same circle.

2. The side A E is common to the two triangles B A E and A C E.

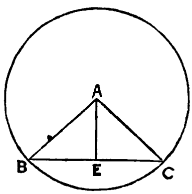
3. Wherefore the two sides B A, A E of the triangle B A E are equal to the two sides C A, A E of the triangle C A E.

4. And the angle B E A is equal to the angle C E A, because they are right angles (Def. 10).

5. Therefore, by the Fourth Proposition, the two triangles A B E and C A E are equal in every respect. Wherefore B E is equal to E C. Q.E.D.

The error in the above demonstration is explained on the following page. The learner will do well not to read the explanation of the mistake till he has first tried to find it out himself.

The error is this, that the angles B E A and C E A are not the angles *contained* by the sides of the triangles which are equal, each to each.



The Fourth Proposition proves that two triangles are equal in every respect, if two sides of the one are equal to two sides of the other, each to each, and if the *angles contained by those sides are also equal.*

Now, the *sides* (see the paragraph marked 3 of the above incorrect solution) are B A, A E of the triangle B A E, and C A, A E of the triangle C A E, and the angles contained by these sides are B A E and C A E respectively; and these angles are *not* given equal (the angles given equal are A E B and A E C), and therefore the triangles cannot be proved equal by the Fourth Proposition. In fact, the equality of B E and C E has to be proved by a subsequent Proposition, namely, the Twenty-sixth.

This concludes the exercises on the Fourth Proposition; and we now pass on to the Fifth, a proposition which bears a name which makes it dreaded by beginners. Those, however, who have come thus far through this treatise need not fear it, for the simple reason that they have already done it. The foregoing Fifth and Sixth Exercises, with the appended deduction from them, is really the Fifth Proposition in a slightly disguised form.

THE FIFTH PROPOSITION DISCUSSED.

The enunciation of the Fifth Proposition may be put thus :

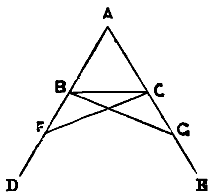
It is given that the two sides of a triangle are equal, in other words, that the triangle is isosceles :—

It is required to prove that the angles at the base are equal.

And if the two equal sides of the isosceles triangle be produced :—

It is required to prove that the angles on the other side of the base are equal.

Particular Enunciation.



Let A B C be an isosceles triangle, having the side A B equal to the side A C :—

It is required to prove that the angle A B C is equal to the angle A C B.

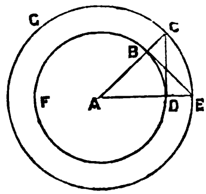
Also, if the equal sides A B, A C be produced beyond B and C, say to the points D and E :—

It is required to prove that the angle D B C is equal to the angle E C B.

In order to prove this proposition, Euclid makes a couple of triangles—which he will be able to compare and prove equal in every respect by the Fourth Proposition—in the following manner.

In B D he takes any point F, and from A E, the greater [*i.e.* greater than A F], he cuts off, by the Third Proposition, a part A G equal to A F, and he joins F C and B G.

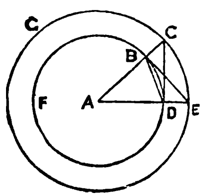
Now, if the learner turns back to Exercise V., he will recognise in the two triangles A D C and A B E a pair of triangles exactly corresponding to the two triangles A B G and A C F (page 79).



And these triangles can be proved equal in every respect, as the two triangles A B E and A D C were proved equal in Exercise V.

This is the first part of Prop. V.

Next, he can prove the triangles B C F and C B G (fig. p. 79) equal in every respect by Prop. IV., in the same manner as the triangles D B E and B D C were proved equal in every respect in Exercise VI.



This is the second part of Prop. V.

Lastly, he can deduce that the angle A B C is equal to the angle A C B (fig. p. 79), as A B D was proved equal to A D B in the deduction appended to the Fifth and Sixth Exercises.

This is the third part of Prop. V.

.It will be a great assistance to the learner to notice particularly that the Fifth Proposition falls naturally into three parts, corresponding, respectively, to Exercises V. and VI., and to the deduction appended to those exercises.

And now, with these hints, and references to these foregoing exercises, the learner is recommended boldly to try to write out the demonstration of the Fifth Proposition without professedly learning it. Let him carefully study Exercises V. and VI., and the appended deduction, also the hints given above; and then write out, if he can, the Fifth Proposition as an exercise on the Fourth.

It is almost too much to hope that he will succeed in the endeavour, but it is worth trying, and if he succeeds it will be a great success. However, let him not be

discouraged if he fails in his attempt to write out the demonstration without professedly learning the proposition.

If he fails to attain this 'great success,' let him read carefully what follows, viz., the proposition written out in due form. If, after doing so, he finds that he is able to write out the demonstration, even this will be a very satisfactory amount of success, for many a poor fellow has spent weeks, if not months, over this proposition, and at the end of the time his mind has been in a state of utter confusion and bewilderment about it.

THE FIFTH PROPOSITION WRITTEN OUT.

PROPOSITION V. THEOREM.

General Enunciation.

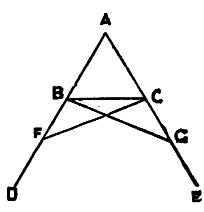
It is given that a triangle is isosceles:—

It is required to prove that the angles at the base are equal.

And if the equal sides be produced:—

It is required to prove that the angles on the other side of the base are equal.

Particular Enunciation.



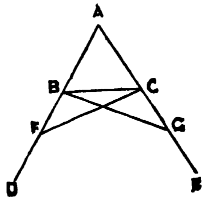
Let A B C be an isosceles triangle, having the side A B equal to A C:—

It is required to prove that the angle A B C is equal to the angle A C B.

And if the equal sides A B and A C be produced to D and E:—

It is required to prove that the angle D B C is equal to the angle E C B.

Construction.



The sides A B and A C being produced to D and E, in B D take any point F, and from A E the greater cut off a part A G equal to A F the less, and join F C, B G.

Demonstration.

PART I.

First, in the two triangles A B G and A C F:
We have :—

1. A G equal to A F (construction).
2. A B equal to A C (hypothesis).
3. Wherefore the two sides F A, A C of the triangle F A C are equal to the two sides G A, A B of the triangle G A B, each to each.
4. And the included angle at A is common to the two triangles F A C and B A G.
5. Therefore, by Prop. IV., the two triangles F A C and B A G are equal in every respect. Wherefore the base B G is equal to the base F C, and the remaining angles, to which the equal sides are opposite, are equal, viz. the angle A B G to the angle A C F, and the angle A G B to the angle A F C.

PART II.

Secondly, in the two triangles F B C and G C B:
We have:

1. The side F C equal to the side B G. It has been proved so above.

2. The side F B is equal to the side G C, as may thus be shown:—The whole A F is equal to the whole A G (cons.), parts of which, namely, A B and A C, are also equal (hyp.). Therefore the remainders F B and G C are equal (Ax. 3).

3. Wherefore the two sides C F, F B of the triangle F C B are equal to the two sides B G, G C of the triangle G B C, each to each.

4. And the included angle B F C has been proved to be equal to the included angle C G B (Part I.).

5. Therefore, by Prop. IV., the two triangles F C B and G B C are equal in every respect. Wherefore the remaining angles, to which the equal sides are opposite, are equal, viz. the angle F B C to the angle G C B, and the angle B C F to the angle C B G.

PART III.

1. Now the whole angle A B G was proved, in Part I., to be equal to the whole angle A C F.

2. Parts of which, viz. the angle C B G, and the angle B C F, were proved equal, in Part II.

3. Therefore the remaining angle A B C is equal to the remaining angle A C B (Ax. 3). And these are the angles at the base of the isosceles triangle.

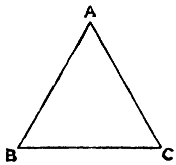
4. Also the angle F B C has been shown (Part II.) to be equal to the angle G C B. And these are the angles on the other side of the base.

Wherefore, if a triangle is isosceles, the angles at the base are equal; and if the equal sides be produced, the angles on the other side of the base are also equal. Q.E.D.

From the above proposition may easily be deduced the Corollary[1] that every equilateral triangle is also equiangular.

Let A B C be an equilateral triangle:—

It is required to prove that it is equiangular.



Demonstration of Corollary.

1. Any two of its sides (as A B and A C) are equal; therefore, by Prop. V. the angle A B C is equal to the angle A C B.

2. Also, any other two of its sides (as A B and B C) are equal; therefore, by Prop. V., the angle B A C is equal to the angle B C A.

3. Also the two other sides, A C, C B, are equal; therefore, again by Prop. V., the angles C A B and C B A are equal.

Therefore all the angles are equal. Q.E.D.

[Of course, having proved that any two angles of the equilateral triangle are, each of them, equal to the third angle, as is done in 1 and 2, it might be inferred by Axiom 1 that they are equal to one another.]

[1] Whatever may be obviously gathered or deduced from a proposition is called a '*corollary*' to it. *Corollarium* meant originally a garland or wreath of thin metal given as a reward. Perhaps the idea of a wreath or garland, hanging from its support, caused the word to be used later, by philosophical writers, to express a deduction, an inference.

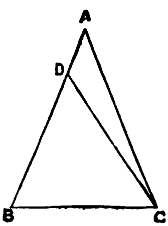
THE SIXTH PROPOSITION DISCUSSED.

The general enunciation of the Sixth Proposition is as follows:

It is given that two angles of a triangle are equal to each other:—

It is required to prove that the sides also which subtend or are opposite to the equal angles, are equal to one another.

Particular Enunciation.



Let A B C be a triangle having the angle A B C equal to the angle A C B:

It is required to prove that the side A B (opposite to the angle A C B) is equal to the side A C (opposite to the angle A B C).

This proposition is what is called a *converse* proposition. It is the converse of the Fifth. By writing out the enunciations of the two propositions, the Fifth and Sixth, side by side, you will see the meaning of the word 'converse.'

In the Fifth Proposition it is said:

If the side A B is equal to the side A C:—the angle A B C is equal to the angle A C B.

In the Sixth Proposition it is said:

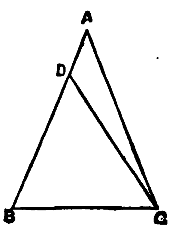
If the angle A B C is equal to the angle A C B:—the side A B is equal to the side A C.

What is given in the Fifth has to be proved in the Sixth, and what is given in the Sixth has to be proved in the Fifth —in other words, the premises (what is given) and the conclusion (what has to be proved) change places.

Now very few converse propositions admit of a direct proof. Euclid cannot prove directly that, if the angle A B C is equal to the angle A C B, the side A C is equal to the side A B. But he *can* and *does* prove that it is *false* to say that A C and A B are unequal; and hence he infers that they are equal.

This method of indirect proof is called a *Reductio ad absurdum.*

To appreciate this method of proof, let us consider Euclid as forcing conviction on an antagonist who denies what Euclid contends is true.



Imagine this antagonist to say: 'I deny that A B is equal to A C.'

Euclid replies: 'If it is *as you say*—i.e., if AB is not equal to A C—one of them must be greater than the other.'

The opponent is obliged to admit this.

'Well, then,' says Euclid, 'suppose it is A B that is the greater, then we can, by Prop. III., cut off from A B, the greater, a part equal to A C the less.'

The opponent must admit this also.

'Suppose it done,' says Euclid, 'that is, suppose B D to be cut off from B A,—which *you say* is the greater,—equal to A C the less.'

'Granted,' says the opponent.

'Then,' adds Euclid, 'we will join C D.'

The opponent must allow him to do so: it is the first postulate.

'Now then,' says Euclid, 'I have before me two triangles A C B and D B C, the bases of which are A B and D C respectively; and these triangles I can prove equal in every respect by the Fourth Proposition, as follows:

1. The side D B of the triangle D B C has been cut off from A B [which the antagonist said was the greater] so as to be equal to the side A C of the triangle A C B.

2. The side B C is common to both triangles.

3. Wherefore the two sides D B, B C of the triangle D B C are equal to the two sides A C, C B, of the triangle A C B.

4. And the angle D B C, contained by the two sides D B, B C, is, *by the hypothesis*, equal to the angle A C B contained by the two sides A C, C B.

5. Therefore, by Prop. IV., the two triangles A B C and D B C are equal in every respect, and therefore in respect of area. That is, the space enclosed by the three straight lines D B, B C, C D, is equal to the space enclosed by the three straight lines A B, B C, C A. But this first space is only a part of the second, and a whole is greater than a part (Axiom 9) ; therefore the conclusion that they are equal is absurd.'

[It is to be hoped that no one who has followed Euclid's reasoning so far will here be disposed to make the common, but weak observation, What is the use of proving what is absurd ? but will rather see that Euclid has now completely 'shut up' his opponent. The opponent had said A B and A C are unequal. Euclid answers, For the sake of argument, suppose it to be as you say, and see what comes of it. And what *does* come of it ? Why this, that if what the opponent asserts, viz., that A B and A C are unequal, is true, a part can be proved equal to a whole. But a part cannot equal a whole, therefore what the opponent asserts, viz., that A B, A C are unequal, is false,[1] and all that Euclid has to add, is :]

'Therefore A B is not unequal to A C, that is, it is equal to it. Which was to be proved.'

This illustration of Euclid's reasoning, when 'mise en scène' in the following manner, has amused, and while

[1] An absurd conclusion must result, either from the hypothesis (the foundation of the reasoning) being false, or from a flaw in the reasoning itself. It is hoped that the pupil's knowledge of the Fourth Proposition, and his power of rightly applying it, are now so confirmed that he will without hesitation decide :—'It is the hypothesis that is false.'

amusing has refreshed and enlightened, boys engaged on the sixth proposition.

It may seem to *some*, perhaps, that the introduction of such a scene is trifling with a grave subject—not to those who have learned by experience what a bar 'rigidity' is to a young geometrician's enjoyment of a Euclid lesson.

The scene is laid at the school of Euclid in the city of Alexandria. The time about 250 years B.C.

Dramatis personæ: the philosopher and a stranger.

[*When the curtain is supposed to rise, a philosopher is seen in profound study. Enter a stranger, who looks about with curious admiration, and then approaches the philosopher.*]

Stranger. Learned philosopher, is this the school of the renowned Euclid?

Philosopher. It is the school of Euclid.

Stranger. I have come from an island in the western ocean, named Britannia, that I may behold and converse with a philosopher, the fame of whose learning has reached to our distant shores. May I see him?

Philosopher. I am Euclid.

Strang. Great and learned philosopher, the last of your propositions which has come to our knowledge in the far west is the fifth, in which you prove that the angles at the base of an isosceles triangle are equal, and I cannot withhold from you my admiration of the lucid, direct, and convincing proof that you have given that this, your discovery, is true. May I ask if you have made any fresh discovery since that which has been made known to us?

Euclid. I have.

Strang. May I ask what is it?

Euc. It is that if two angles of a triangle are equal, the sides opposite to those angles are also equal.

Strang. And, revered philosopher, can you give a proof that this is true, as lucid, direct, and convincing as

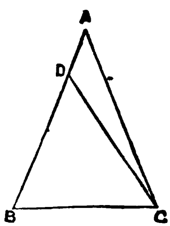
that which you gave that if the sides of a triangle are equal, the angles at the base of the triangle are also equal?

Euc. I cannot.

Strang. Then, O learned Euclid, great as is your reputation, I withhold my assent to the assertion you make. Taught by you, I refuse to give assent to that which has not been demonstrated.

Euc. But though I cannot prove directly that the sides are equal, I can prove that it is false to say that they are unequal.

Strang. Ah! How so?



[*Euclid here draws the triangle A B C.*]

Euc. If the sides B A, A C, are not equal, stranger, what must they be?

Strang. They must be unequal.

Euc. And if they are unequal, one of them must be what . . . ?

Strang. One of them must be greater than the other.

Euc. And if one of them (say A B) is greater than the other, what may be cut off from the greater?

Strang. A part may be cut off equal to the less. We were taught how to do so in your third proposition.

Euc. And if that be done—if B D be cut off from B A equal to A C, may the points D and C be joined?

Strang. They may be joined. In your first postulate you asked that you might be allowed to do so.

Euc. Now, stranger, fix your attention on the two triangles A C B and D B C, and tell me, of these triangles, what straight lines are the bases?

Strang. Clearly D C and A B are the bases, these straight lines being opposite to the angles A B C and A C B, which are given equal.

Euc. And what do you observe of the side B C?

Strang. That it is a side common to both triangles.

Euc. Good! So that you have the two sides D B, B C of the triangle D B C . . . ?

Strang. Equal to the two sides A C, C B of the triangle A C B.

Euc. Right. And what do you know of the included angles D B C and A C B?

Strang. I know that they are equal. It was so given in the enunciation.

Euc. What then can you say, by Prop. 4, of the two triangles D B C and A C B?

Strang. I can say that they are equal in every respect.

Euc. And if the triangles D B C and A C B are equal in *every* respect, what follows with respect to the areas D B C and A C B?

Strang. It follows that they are equal.

Euc. But can the areas A B C and D B C be equal?

Strang. They cannot. For the area D B C is but a part of the area A C B, and your ninth axiom says that a whole is greater than its part.

Euc. But *on the supposition* that A B and A C are unequal, what has been proved.

Strang. It has been proved that those areas are eqnal.

Euc. What then follows with regard to the supposition on which it was proved that the areas B A C and B D C are equal.

Strang. It follows that it was a false supposition.

Euc. And if it is false to say that B A and A C are equal, what must they be?

Strang. They must be equal. Great Euclid, I am convinced of the truth of your discovery. I return to my country penetrated with new admiration of your wisdom. I will even undertake to prophesy that when more than two thousand years have elapsed, the youth of Britannia will still be invigorated by the study of the discoveries which you have made, and of the reasoning by which you have proved the truth of those discoveries.

Illustrious philosopher, farewell. [*Exit* Stranger.

Perhaps some learners will be able, with the above elucidation of Euclid's reasoning, to write out the proposition as is done below, without reading further for the present.

THE SIXTH PROPOSITION WRITTEN OUT.

Proposition VI. Theorem.

General Enunciation.

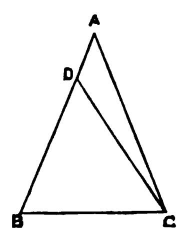
It is given that two angles of a triangle are equal to each other :—

It is required to prove that the sides also which subtend, or are opposite to, the equal angles are also equal.

Particular Enunciation.

Let A B C be a triangle having the angle A B C equal to the angle A C B :—

It is required to prove that the side A C is equal to the side A B.



Construction.

If A B be not equal to A C,

One of them must be greater than the other.

Suppose A B to be the greater.

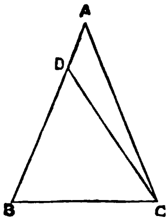
Then from A B cut off a part B D equal to A C the less (Prop. 3).

And join D C.

Demonstration.

Then in the triangles D B C and A C B we have :—

1. D B equal to A C (they were made so on the assumption that A B and A C were unequal).



2. B C is common to both triangles.

3. Wherefore the two sides A C, C B of the triangle A C B, are equal to the two sides D B, B C of the triangle D B C.

4. And the included angle A C B is by the hypothesis equal to the included angle D B C.

5. Therefore, by Prop. IV., the two triangles D B C and A C B are equal in every respect. Wherefore they are equal in respect of area; that is, the enclosed space D B C is equal to the enclosed space A C B. But D B C is only a part of A B C, and therefore cannot be equal to it (Ax. 9).

Therefore A C is not unequal to A B, that is, it is equal to it.

Wherefore, if two angles of a triangle be equal to one another, the sides also which are opposite to the equal angles are equal. Q.E.D.

Corollary. Hence it may be deduced that every equiangular triangle is also equilateral.

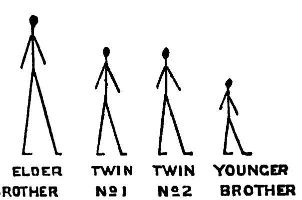
The truth of this corollary is deduced from the Sixth Proposition precisely as the former corollary (the converse of this) was deduced from the Fifth Proposition.

THE SEVENTH AND EIGHTH PROPOSITIONS DISCUSSED.

These two propositions must be taken together. They must be read, to use an expression of Sir Robert Peel's, 'without solution of continuity.' In fact in the following discussion, the two are mixed together, and made *one* proposition.

We will begin the discussion with a story, the use of which will be seen by-and-by.

There is a family of four brothers; the two middle ones are twins, the counterparts of each other, both being of exactly equal height. We will call the twins No. One and No. Two. There is an elder brother who is known to be taller than twin No. One, and there is a younger brother who is known to be shorter than twin No. Two. Euclid wants to prove that the elder brother is taller than the younger brother. He does it in this way:



We know, he says, that twin No. Two is taller than the younger brother.

But twin No. One is equal to twin No. Two.

Therefore twin No. One is taller than the younger brother.

But the elder brother is taller than twin No. One.

Much more, he concludes, the elder brother is taller than the younger.

In a school in which the writer of this treatise takes much interest, a modification of the above story is in vogue, which, by the dash of comic introduced in it, and

by its dealing with known persons, helps pleasantly to fix the argument in the boys' minds.

In the school in question there are twin-brothers, of equal height; we will call them John and Edward Swayne. There is also a most respected assistant-master, whom we will call Mr. Davis, very tall. He is known to be taller than John Swayne. Lastly, there is a remarkably small boy, who may be called Douglas White. He is known to be shorter than Edward Swayne. And the joke is to prove that Mr. Davis is taller than Douglas White, as follows :—

1. Edward Swayne is taller than Douglas White.

2. But Edward Swayne is of the same height as his brother John.

3. Therefore John Swayne is taller than Douglas White.

4. But Mr. Davis is taller than John Swayne.

5. Much more, then, is Mr. Davis taller than Douglas White. Q.E.D.

Other schools will supply like illustrations.

The enunciation of the Eighth Proposition is as follows :—

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal :—It is required to prove that the angle contained by the two sides of the one will be equal to the angle contained by the two sides equal to them of the other.

Having read the above enunciation, the learner will do well to compare it with that of the Fourth. He will find that in both propositions the two first conditions are alike: namely that the two sides of one triangle are equal to the two sides of the other, each to each; and that whereas in the Fourth Proposition the third condition is that the included angles are equal, and it is required to prove that

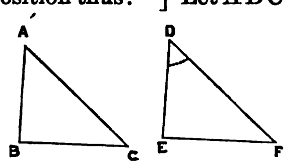
the bases are equal, in the Eighth Proposition it is given, as the third condition, that the bases are equal, and it is required to prove that the included angles are equal.

Like the Fourth Proposition, the Eighth is a very important one. The Seventh is only a subsidiary proposition, introduced in order to prove the Eighth.

The proof which Euclid gives of the Eighth is akin to that which he gave of the Fourth.

He proved in the Fourth that the bases were equal by showing that, if one triangle were applied to the other, as directed, the bases would fill the same space; so, in the Eighth, he proves that, if one triangle be applied to the other, as he directs, the included angles will fill the same space.

[Euclid begins the Eighth Proposition thus:—] Let A B C and D E F be two triangles which have the two sides B A, A C of the one equal to the two sides E D, D F of the other, each to each, viz., B A to E D, and A C to D F, and likewise the base B C equal to the base E F:—



It is required to prove that the included angles B A C and E D F are equal.

[He proceeds as follows:—]

Let one of these triangles (A B C) be applied to the other (D E F), so that the point B may lie on the point E; and that the base B C may lie, as far as it goes, along the base E F; then will the point C fall on the point F, because B C is equal to E F; and in fact the two bases B C and E F will occupy the same space.

[And this being so:—the next question will be, where will B A and A C lie? will they lie on E D and D F? If they do, the angles B A C and E D F will fill the same space, and therefore be equal (Ax. 8), which is what has to be proved.

In fact, to recur once more to the curved rim (see page 6), if B A, A C lie on E D, D F, the curved rim drawn to mark the angle B A C, and that drawn to mark the angle E D F, will manifestly be drawn through the same space (see fig. page 93).

Now Euclid cannot prove that, B C lying on E F, B A will lie on E D, and C A on D F, as he did in the Fourth Proposition, because the angle A B C has not been given equal to the angle D E F, nor has the angle A C B been given equal to the angle D F E.

He is obliged, therefore, to reason in a different way.

His reasoning is as follows :—]

The bases B C and E F now occupying the same space, if the two sides B A, A C of the triangle A B C do not lie along the two sides E D, D F, of the triangle D E F, they must take some other direction; in which case one of three things must happen :—

Either (i) the vertex of each triangle will lie outside the other triangle, as in fig. 1.

Or (ii) the vertex of one of the triangles will lie inside the other triangle, as in fig. 2.

Or (iii) the vertex of one of the triangles will lie on a side of the other triangle, as in fig. 3.

Fig. 1.

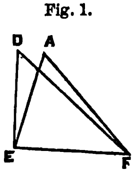


Fig. 2.

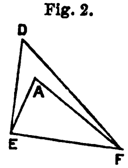
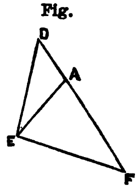


Fig.



[And if it can be shown that it is impossible that either of the vertices can lie in any one of the three positions above named, then it will follow that the two sides B A, A C must lie along the two sides E D, D F.

Now, in order to show that it is impossible that either

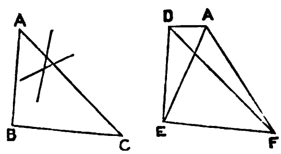
of the vertices can occupy any one of the above-named positions, Euclid takes each of the three cases, one after another, and examines them; his reasoning is as follows:—]

Case 1.—First suppose that each of the vertices falls outside the other triangle,[1] and let us see what comes of the supposition.

Construction.

Join the vertices A and D.

[Observe because E D is equal to E A (hyp.), we know that the angle E D A is equal to E A D (Prop. 5).



Here, then, are the twins, of the story given above, Edward and John Swayne; let E D A be Edward and E A D be John.

Now, the angle E D A (Edward Swayne) is greater than the angle F D A (Ax. 9). The angle F D A then is Douglas White.

And the angle F A D is greater than the angle E A D (John Swayne, the other twin); F A D then is Mr. Davis.]

Demonstration.

1. Since the side E A is equal to the side E D, the angle E D A [Edward Swayne] is equal to the angle E A D [John Swayne] (Prop. 5).

2. But the angle E D A [Edward Swayne] is greater than the angle F D A [Douglas White]. (Ax. 9.)

3. Therefore also the angle E A D [John Swayne] is greater than the angle F D A [Douglas White].

4. But the angle F A D (Mr. Davis) is greater than the angle E A D [John Swayne]. (Ax. 9.)

[1] The strokes drawn across the triangle A B C are intended to remind the learner that the triangle A B C is no longer there, but is now laid, as was directed, on the triangle D E F. [See page 93.]

5. Much more is the angle F A D [Mr. Davis] greater than the angle F D A [Douglas White].

[Thus Euclid has shown that because A E is equal to D E, the angle F A D must be greater than F D A.

He continues the reasoning thus:—]

Again, because F A is equal to F D, the angle F A D is equal to the angle F D A (Prop. V.).

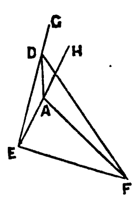
That is, if each of the vertices falls outside of the other triangle, the same angles (F A D and F D A) are equal and unequal at the same time, which is absurd.

Therefore it is impossible that each vertex can fall *outside of* the other triangle.

Case 2.—Can one of the vertices fall inside the other triangle? Let us suppose that one of them (as A) does so, and let us see what comes of this supposition.

Construction.

In this case, besides joining D A, produce the sides E D, E A to G and H.



[Now because E D is equal to E A, the angles G D A and H A D, on the other side of the base A D, are equal (Prop. 5).

These angles then (G D A and H A D) are our twins (E. and J. Swayne), and clearly, as in the former case, the angle F D A is Douglas White, and the angle F A D is Mr. Davis.]

Euclid's reasoning in this, the second case, is as follows[1]:—

[1] The learner is earnestly advised, before reading further, to say, or write out, the five steps of the reasoning by which Euclid proves, in the second case, that the angle F A D is greater than F D A. As an incitement to do so, he may be told that many learners, after spending days, and even weeks, over this proposition, have entreated that they might be excused learning the second case, and have begged that it might be the first case that should be set in the examination! The five steps are given in the text.

Demonstration.

1. Because E D is equal to E A, the angle G D A is equal to the angle H A D (Prop. 5).

2. But the angle G D A is greater than the angle F D A.

3. Therefore also the angle H A D is greater than the angle F D A.

4. But the angle F A D is greater than the angle H A D.

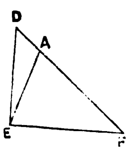
5. Much more then the angle F A D is greater than the angle F D A.

6. Again, because F A is equal to F D, the angle F A D is equal to the angle F D A.

That is, if either of the vertices falls inside the other triangle, the same angles (F A D, and F D A) are equal and unequal at the same time; which is absurd.

Therefore it is impossible that one of the vertices can fall *inside* the other triangle.

The third case, viz., that the vertex A does not fall *on the side* of the triangle D E F, needs no demonstration, for it is evident from Ax. 9 that F A is not equal to F D.



[The conclusion from this reasoning is as follows :—

Since it is impossible that the vertex of each triangle can lie outside the other triangle, or that the vertex of one of the triangles can lie within the other triangle, or on the side of it :—] Therefore the two straight lines B A, A C must lie along E D, D F, and therefore (as explained above, pages 93, 94) the angles B A C and E D F fill the same space, and therefore they are equal (Ax. 8). Q. E. D.

The writer finds that by dealing with the seventh and eighth Propositions in the foregoing way—mingling them together, and not introducing the reasoning of the seventh Proposition till the need of it is felt—the learner is enabled more easily to see what Euclid is aiming at, and to appreciate his reasoning.

If before passing on to Euclid's seventh and eighth Propositions formally written out, the learner would like to test whether he has got hold of Euclid's reasoning, let him write out these Propositions as they have been given, mingled together, in the foregoing discussion, beginning page 93, 'Let A B C,' &c., leaving out all that is contained between [ ].

Pupils who have gone through the propositions in this way, have really done Euclid's seventh and eighth Propositions. So that if it is thought that they have spent enough time for the present on these two Propositions, they may proceed at once to page 104, and so on to the ninth Proposition; leaving to a future time the writing out the two Propositions formally, as given in Euclid.

When Euclid's argument has been fully grasped, there will be no difficulty in dividing the Proposition into two parts, according to Euclid's method, with the aid of the following explanation.

Euclid proves first, in the seventh Proposition, that on the same base and on the same side of it there cannot be two triangles, having the two sides ending in one extremity of the base equal to each other, and likewise the two sides ending in the other extremity.

This statement of Euclid's is equivalent to saying:— If on the same base and on the same side of it, there be two triangles, having the two sides ending in one extremity of it equal to each other, and likewise the two sides ending in the other extremity, then must the sides ending in each extremity of the base, each lie one on the other—so that (if the reader will pardon the expression) the two triangles standing on the same base are not two triangles, but only one and the same triangle.

Having proved this, Euclid enunciates the eighth Proposition (see page 92), and when, according to his method of superposition, he has brought the two bases

together, and shown that they occupy the same space; then he has before him exactly the case contemplated in Proposition VII., viz., two triangles on the same base, and on the same side of it, having the two sides ending in one extremity of the base equal to each other, and likewise the two sides ending in the other extremity.

But in this case, as proved, Prop. VII., the two sides ending in each extremity of the base must lie one on the other [for otherwise there would be two triangles], and therefore the angles contained by them fill the same space, and are therefore equal (Ax. 8), which was to be proved in Prop. VIII.

Euclid's two Propositions, the seventh and eighth, formally written out, are given on the following pages.

THE SEVENTH PROPOSITION WRITTEN OUT.

Proposition VII. Theorem.

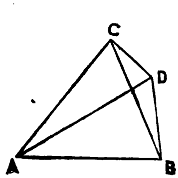
General Enunciation.

It is required to prove that :—

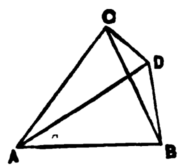
On the same base, and on the same side of it, there cannot be two triangles which have the two sides terminated in one extremity of the base equal to each other, and likewise the two sides terminated in the other extremity.

Particular Enunciation.

On the same base A B and on the same side of it, there cannot be two triangles which have the two sides A C and A D, terminated in the extremity A, equal to each other, and *likewise* the two sides B C and B D terminated in the extremity B.



If it be possible, the vertex of each triangle must either fall without the other triangle; or the vertex of one of them must fall within the other; or the vertex of one of them must fall on the side of the other.



First suppose the vertex of each triangle to fall without the other.

Construction.

Join the two vertices C and D.

Demonstration.

1. Then because A C is equal to A D, the angle A C D is equal to the angle A D C (Prop. 5).

2. But the angle A C D is greater than the angle B C D (Ax. 9).

3. Therefore also the angle A D C is greater than the angle B C D.

4. Much more is the angle B D C greater than the angle B C D.

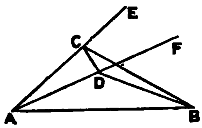
5. On the other hand, because B D is equal to B C, the angle B D C is equal to the angle B C D (Prop. 5).

6. That is, if the vertex of each triangle is without the other, the same angles B C D and B D C are equal and unequal at the same time, which is impossible; therefore the vertex of each triangle does not fall without the other.

Secondly, suppose that the vertex D, of one of the triangles (say A D B), falls within the other triangle (A C B).

Construction.

Besides joining the vertices C, D, produce the two sides A C and A D to E and F.



Demonstration.

1. Then because A C is equal to A D, the angle E C D is equal to the angle F D C (Prop. 5).

2. But the angle E C D is greater than the angle B C D (Ax. 9).

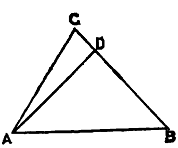
3. Therefore also the angle F D C is greater than the angle B C D.

4. Much more is the angle B D C greater than the angle B C D.

5. On the other hand, because B C is equal to B D, the angle B D C is equal to the angle B C D (Prop. 5).

6. That is, if the vertex of one of the triangles falls within the other, the same angles B C D, B D C, are equal and unequal at the same time, which is impossible; therefore the vertex of one of the triangles does not fall within the other.

The case in which the vertex D of one of the triangles A D B is on the side of the other triangle A C B, needs no demonstration, for it is evident from Ax. 9 that B C cannot be equal to B D.



Therefore on the same base and on the same side of it there cannot be two triangles which have the two sides terminated in one extremity of the base equal to each other, and likewise the two sides terminated in the other extremity. Q.E.D.

From the Seventh Proposition, Euclid immediately deduces the Eighth. After the foregoing discussion and explanation, it will be sufficient to write it out formally, without any further remarks.

THE EIGHTH PROPOSITION WRITTEN OUT.

PROPOSITION VIII. THEOREM.

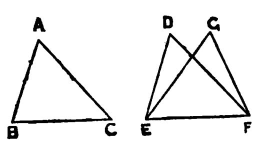
General Enunciation.

It is given that two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal:—

It is required to prove that the angle contained by the two sides of one of the triangles is equal to the angle contained by the two sides equal to them of the other.

Particular Enunciation.

Let A B C and D E F be two triangles which have the two sides B A, A C of the one equal



to the two sides E D, D F of the other, each to each, viz. B A to E D, and A C to D F; and have likewise the base B C equal to the base E F :—

It is required to prove that the included angle B A C is equal to the included angle E D F.

Construction.

Let the triangle A B C be applied to the triangle D E F, so that the point B may be on E, and the base B C may lie along E F.

Demonstration.

Then shall the point C come on the point F, because B C is equal to E F; and the bases B C, EF will occupy the same space.

This being the case, the two sides B A, A C must lie along the two sides E D, D F, for if not they will have a different situation as E G, G F.

And then, on the same base and on the same side of it, there *will* be two triangles which have the two sides terminated in one extremity of the base equal to each other, and likewise the two sides terminated in the other extremity.

But this is impossible (Prop. 7).

Therefore the two sides B A, A C cannot but lie on the two sides E D, D F, and therefore the angles B A C and E D F fill the same space, and are equal (Ax. 8).

Wherefore, if two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; then shall the angle contained by the two sides of the one be equal to the angle contained by the two sides equal to them of the other. Q.E.D.

Euclid might have proceeded to show that the two remaining angles are also equal, as well as the included areas, for he has here shown, as completely as he did in the Fourth Proposition, that the two remaining angles and the included areas respectively *fill the same spaces*. He does

not, however, proceed any further, nor is it necessary, for when he has proved that the included angles are equal, he has before him two triangles which have two sides of the one equal to two sides of the other, and likewise the included angles equal: and therefore, by the Fourth Proposition, the two triangles are equal in every respect.

The learner, having proved the Eighth Proposition as well as the Fourth, can now go forward with these two propositions, one in either hand, engines, as it were, of use and power in the solution of other problems and theorems.

The next four propositions are problems, to be solved by one or other of these two propositions. But the learner must be careful to apply them rightly. If it is the included angles which have to be *proved* equal, he must apply the Eighth Proposition. If it is the bases which have to be *proved* equal, he must apply the Fourth Proposition.

In both cases the first three of the five steps of reasoning, often repeated in the exercises on the Fourth Proposition, are the same, whether it is the Fourth or the Eighth Proposition which is applied. The last two steps change places.

In applying the Fourth Proposition, the fourth and the fifth steps of the reasoning are,

(4) And the included angles [naming them] are equal [the reason why these angles are known to be equal being added]. And the conclusion is:

(5) Therefore by the Fourth Proposition the bases are equal. Q.E.D.

In applying the Eighth Proposition, the last two steps of the reasoning are as follows:

(4) And the bases are equal [the reason why they are known to be equal being added]. And the conclusion is:

(5) Therefore, by the Eighth Proposition, the included angles are equal. Q.E.D.

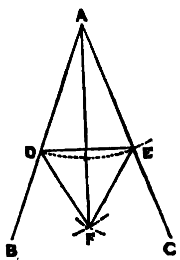
THE NINTH PROPOSITION DISCUSSED.

With these preliminary remarks we turn to the Ninth Proposition, the enunciation of which is as follows:

Given a rectilineal angle :—

It is required to bisect it.

Euclid draws an angle and calls it the angle B A C, and adds, It is required to bisect B A C.



His construction is as follows:

In A B he takes any point D.

And by the Third Proposition he cuts off from A C a part A E equal to A D.

He joins D E.

And on the side of D E, remote from A, he describes an equilateral triangle D F E by Prop. I.

[You will be told by and by why he says on the side of D E *remote* from A.]

He adds, Join A F.

And he concludes thus:

Then will the angle B A C be bisected by the straight line A F.

Before exhibiting to the learner the demonstration of this proposition in a formal manner, the following suggestions are given, to encourage him to make out the demonstration for himself.

1. We have to prove that the angle B A C is bisected. That is, we have to prove that the angle B A F is equal to the angle C A F. (See fig. on page 107.)

2. As it is 'included angles' and *not* 'bases' that have to be proved equal, it is the Eighth Proposition that must be used.

3. *Probably* the two triangles to be compared by the Eighth Proposition are A D F and A E F, and we shall know that they *certainly* are, if we can show that these triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the bases equal.

4. Now which are the bases? Clearly D F and E F, for these are the sides of the triangles A D F and A E F, which are opposite to the angles D A F and E A F which have to be proved equal.

5. And D F, E F being the bases, the two sides of the triangle D A F will be D A and A F, and the two sides of the triangle E A F will be E A and A F. And the angles contained by these sides (or the included angles) will be D A F and E A F.

6. Can we show that the two sides here named of the triangle D A F are equal to the two sides named of the triangle E A F, and that the bases are equal?

For an answer to this question we must look back to the construction.

The learner, calling to mind the observation made page 32, will remember that since in the construction he was told to make A E equal to A D, he is sure to meet in the demonstration 'A E is equal to A D because they were made so.'

Also since in the construction he was told, on D E, to describe an equilateral triangle, he is sure to meet in the demonstration the sentence—that certain straight lines are equal because they are the sides of an equilateral triangle.

With these suggestions the learner is advised, without reading further as yet, to write out the five steps of the demonstration by which it is proved that the included angles

D A F and E A F are equal, in other words, that the angle B A C is bisected by the straight line A F.

It may appear to some learners that this is a very tedious way of getting up the proposition. They may think that it would be much shorter just to read over the demonstration formally written out. In reply to this objection it may be said that these suggestions, which appear to the learner long when written out formally, will, after a little practice, pass through his mind instantaneously, one may say, when he has before him two triangles to be compared; and certainly a little self-reliance and independent thought are of the highest value in mastering the propositions of Euclid.

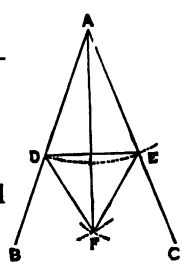
The learner who follows the advice given above, and writes out the demonstration without reading farther at present, may compare his demonstration when he has written it out, with that which is given below.

THE NINTH PROPOSITION WRITTEN OUT.

Proposition IX. Problem.

General Enunciation.

A rectilineal angle being given :—
It is required to bisect it.



Particular Enunciation.

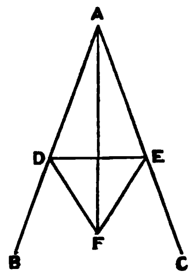
Let B A C be a given rectilineal angle:—
It is required to bisect it.

Construction.

1. In A B take any point D.
2. From A C the greater cut off A E equal to A D the less (Prop. 3).

3. Join D E.

4. On the side of D E remote from A describe an equilateral triangle D F E.



5. Join A F.

Then will the angle B A C be bisected by the straight line A F.

Demonstration.

1. In the two triangles A D F and A E F, the side A E is equal to the side A D (they were made so).

2. The side A F is common to both triangles.

3. Wherefore the two sides D A, A F of the triangle D F A are equal to the two sides E A, A F of the triangle E F A, each to each.

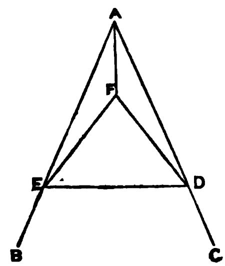
4. And the base D F is equal to the base E F because they are sides of an equilateral triangle.

5. Therefore by the Eighth Proposition the included angle D A F is equal to the included angle E A F, that is, the angle B A C has been divided into two equal angles D A F and E A F: which was to be done.

With regard to Euclid's saying 'On the side of D E *remote* from A, describe the equilateral triangle D F A,' the following explanation will suffice.

The three straight lines D A, A E, E D, might *possibly* be all equal, that is, D A E *might* be an equilateral triangle. In this case if the equilateral triangle D F E were described on the same side of D E on which the point A is, the vertex F of this triangle would fall exactly on A. In this case there would be no straight line A F.

When this is not the case, and the vertex of the equilateral triangle falls within the angle B A C, as in the adjoining figure, it may be a useful exercise to prove that the angle D A F is equal to the angle E A F; the equilateral triangle D E F being described on the same side of the base as that on which the point A is.



THE TENTH PROPOSITION DISCUSSED.

The Tenth Proposition is as follows:

Given a straight line:—

It is required to bisect it.

In the Ninth Proposition Euclid taught us how to bisect *an angle.* He now in the Tenth Proposition shows that, being able to bisect an angle, we can, by doing so, bisect a straight line.

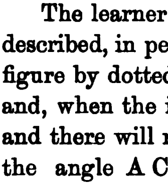
He says, Let A B be a straight line:—

It is required to bisect it.

He begins the construction by describing an equilateral triangle upon A B.

Then, referring us to the Ninth Proposition, he says, Bisect the angle A C B. [*Obs.* He does not say triangle.]

In order to do this, we should have in C A to take any point E, and from C B to cut off C F equal to C E. Then we should have to join E F, and on the side of E F remote[1] from C we should have to describe an equilateral triangle E F G; we should then, by joining C G, bisect the angle A C B.



The learner is advised to do all the construction, here described, in pencil, [the pencil lines are represented in the figure by dotted lines] then let him 'ink in' the line C G, and, when the ink is dry, let the pencil marks be effaced, and there will remain only the straight line C G bisecting the angle A C B. Then let him carefully produce the

[1] This is the case Euclid provides for by saying, in the Ninth Proposition, 'On the side of D E *remote* from A;' for if the figure is carefully drawn you will find (what you will be able to prove hereafter) that A B C being an equilateral triangle, C E F is also equilateral, so that if the triangle E G F were described on the same side of E F as C, the point G would come exactly on C.

bisecting line C G, if necessary, till it intersects the given straight line in D.

Euclid then says, The straight line A B is bisected at the point D.

Suggestive Hints for the Demonstration.[1]

These will be very similar to those given in the Ninth Proposition.

In order to prove that the given straight line A B is bisected in D, we must show that A D is equal to D B; and as it is *bases*, not included angles, which are to be proved equal, it is the Fourth Proposition that has to be used; and most probably the two triangles to be compared are the triangles A C D and B C D.

If so, what are their bases? Clearly A D and D B, these being the straight lines which have to be proved equal; and knowing the bases, we know the two sides, and the angles contained by those sides. All then that is required for the demonstration is to show that the two sides A C and C D of the triangle A C D are equal to the two sides B C, C D of the triangle B C D, each to each, and that the angles contained by these sides are equal.

To find if they are so, the learner will have to look back to the construction. He will find that he is there told on A B to describe an equilateral triangle, therefore he may be sure that in the demonstration he will have to say, of certain straight lines, 'they are equal because they are sides of an equilateral triangle.' Also he was told, by the Ninth Proposition, to bisect the angle A C B, therefore he may be sure that he will have to set down in the demonstration, 'A C D is equal to B C D because the angle A C B was bisected by C D.'

With these hints, or without them, the learner is advised here, without reading further at present, to write down the five steps of the demonstration by which it is proved that

[1] If any learner feels that he does not need these hints, he can pass them over, and without their help write out the five steps of the demonstration. He can then compare his demonstration with that given below, where the Tenth Proposition is fully written out.

A B is bisected in D. When written out, it may be compared with the following.

THE TENTH PROPOSITION WRITTEN OUT.

Proposition X. Problem.

General Enunciation.

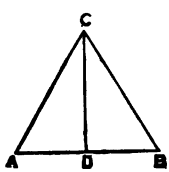
A straight line being given :—
It is required to bisect it.

Particular Enunciation.

Let A B be a given straight line :—
It is required to bisect it.

Construction.

On A B describe the equilateral triangle A B C.



Bisect the angle A C B (Prop. IX.) by the straight line C D intersecting the straight line A B in the point D.

Then will A B be bisected in the point D.

Demonstration.

1. In the two triangles A C D, and B C D, the side A C is equal to the side B C because they are both sides of an equilateral triangle.
2. The side C D is common to both triangles.
3. Wherefore the two sides A C, C D of the triangle A C D are equal to the two sides B C, C D of the triangle B C D, each to each.
4. And the included angle A C D is equal to the included angle B C D, because the angle A C B is bisected by the straight line C D.
5. Therefore by the Fourth Proposition the base A D is equal to the base D B, that is, the straight line A B is bisected in the point D. Q.E.F.

THE ELEVENTH PROPOSITION DISCUSSED.

The enunciation of the Eleventh Proposition is as follows:

Given a straight line and a point in it:—

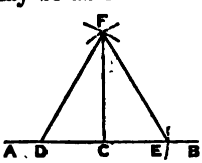
It is required from the given point to draw a straight line at right angles to the given straight line.

Particular Enunciation.

Let A B be the given straight line, and C the given point in it:—

It is required from the point C to draw a straight line at right angles to A B.

Euclid will want, in order that he may be able to solve this problem, two points in the straight line A B, equally distant from C. Therefore, he says, In A C take any point D, and from C B cut off C E equal to C D (Prop. 3).



Then, on the part of the given straight line intercepted between D and E, he describes an equilateral triangle D F E.

And upon joining the vertex F with the given point C, he says that C F will be at right angles to A B.

It will be necessary here to call to mind the tenth definition, in which it is said, When one straight line falling on another straight line makes the adjacent angles equal, each of them is a right angle.

In order then to prove, in conformity with this definition, that C F is at right angles to A B, all that is required is to prove that the angles F C D and F C E are equal; for if they are equal, each of them is a right angle.

With this to guide him, let any learner, who is able, now write out the five steps of the demonstration by which

it is proved that the angle F C D is equal to the angle F C E.

For the help of those who cannot, the hints given in the Ninth Proposition are here repeated. They are here given in the form of suggestive questions, the correct answers to which are given in a foot-note, which it is hoped the learner will not consult till he has written out his own answers to the questions.

1. What have you here to prove equal, 'bases' or 'included angles'?

2. Then what Proposition has to be used. The Fourth or the Eighth?

3. What are the two triangles to be compared?

4. What are their bases?

5. And what therefore are the two sides, and what are the included angles of each of the triangles which have to be compared?

6. In the construction you were told to make C E equal to C D: what sentence then are you sure of having in the demonstration?

7. Also you were told on D E to describe an equilateral triangle: what sentence on this account are you certain to have in the demonstration? [1]

With these hints (if he has not done it without them) the learner is advised to write out the five steps of the demonstration, by which it is proved that the angle F C D is equal to the angle F C E.

When written out it may be compared with the demonstration given on the next page.

[1] *Ans.* 1. Included angles. *Ans.* 2. The eighth. *Ans.* 3. C D F, and C E F. *Ans.* 4. D F and E F, these straight lines being opposite to the angles to be proved equal. *Ans.* 5. D C, C F are the two sides of the triangle D C F, and E C, C F are the two sides of the triangle E C F; the included angles are D C F and E C F. *Ans.* 6. We shall have the sentence 'C E is equal to C D, because it was made so.' *Ans.* 7. We shall have the sentence 'Two straight lines are equal (most probably F D and F E), because they are the sides of an equilateral triangle.'

THE ELEVENTH PROPOSITION WRITTEN OUT.

PROPOSITION XI. PROBLEM.

General Enunciation.

A straight line being given, and a point in it:—

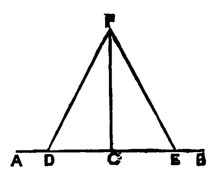
It is required from the given point to draw a straight line at right angles to the given straight line.

Particular Enunciation.

Let A B be a given straight line, and C a given point in it:—

It is required from C to draw a straight line at right angles to A B.

Construction.



In A C take any point D, and from C B cut off C E equal to C D (Prop. III.).

On D E describe the equilateral triangle D F E.

Join F C.

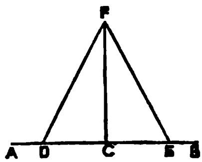
Then shall F C be at right angles to A B.

Demonstration.

1. In the two triangles F C D and F C E, the side C E is equal to the side C D, because it has been made so.
2. The side C F is common to both triangles.
3. Wherefore the two sides D C, C F of the triangle D C F are equal to the two sides E C, C F of the triangle E C F, each to each.
4. And the base D F is equal to the base E F, because they are the sides of an equilateral triangle.

5. Therefore by the Eighth Proposition the included angle D C F is equal to the included angle E C F.

And these are adjacent angles.



6. But when a straight line standing on another makes the adjacent angles equal, each of them is a right angle (Def. 10).

7. Therefore each of the angles D C F and E C F is a right angle, that is, F C is at right angles to A B.

And it is drawn from the point C. Q.E.F.

The following observation, suggested by a valued correspondent, may interest the learner.

It was said (page 5) that, according to Euclid's definition of an angle, when two straight lines meeting at a point were drawn so far asunder, that they were exactly in opposite directions, there ceased to be an angle between them. The learner may now be told that in modern books of Geometry and of Trigonometry, two straight lines in the same straight line, and in opposite directions, are considered to contain an angle equal to two right angles.

Now in the Eleventh Proposition C A and C B (see last figure) may be considered two straight lines, containing an angle equal to two right angles. If the learner, with this hint, will now compare the construction and demonstration of the ninth and eleventh propositions, he will find they are exactly similar.

In fact, to draw a straight line at right angles to a given straight line from a given point in it, is equivalent to bisecting an angle equal to two right angles.

THE TWELFTH PROPOSITION DISCUSSED.

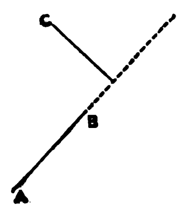
The enunciation of the Twelfth Proposition is as follows:

Given a straight line of unlimited length, also a point without it:—

It is required from the given point to draw a straight line perpendicular to the given straight line.

It will be seen that the Twelfth Proposition differs from the Eleventh only in this, that in the Eleventh the given point is *in* the given straight line and in the Twelfth the given point is *not in* the given straight line, but out of it—anywhere, in fact, *except* in it.

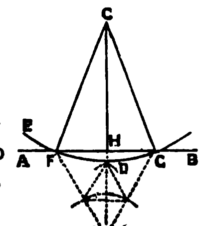
Note also that the words ' of unlimited length ' are introduced into the enunciation of the Twelfth Proposition. The necessity of this is shown in the adjoining figure, where it is clear that if A B were limited (if we were not allowed to produce it), it would be impossible from the point C to draw a straight line perpendicular to A B.



Particular Enunciation.

Let A B be a straight line of unlimited length, and let C be a given point without it.

It is required from the point C to draw a straight line perpendicular to A B.

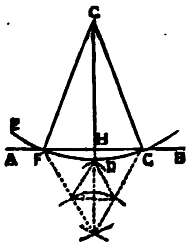


As in the Eleventh Proposition, so in the Twelfth, Euclid will require two points in the given straight line A B equally distant from the given point C.

To get them, he takes any point D on the other side of A B, and by describing the circle E F G,

with the centre C and at the distance C D, he gets the two points (viz. F and G where the circle intersects the straight line A B) equally distant from C.

As a suggestive hint for the next step of the construction, let the learner ask himself whereabouts in F G he thinks the foot of the perpendicular will come. The answer is clearly, midway between F and G. This thought suggests the next step in the construction, which is: bisect F G [1] in the point H by the Tenth Proposition.



Euclid then joins C H and says that C H will be perpendicular to A B.

It will be seen by reference to the Tenth Axiom, that all that is required, in order to show that C H is at right angles (or, in other words, perpendicular) to A B, is to prove that the angle F H C is equal to the angle G H C.

With a view to the demonstration Euclid joins C F and C G.

It is to be hoped that most of the readers of this treatise will at once, without further help, write off the five steps of the demonstration by which it is proved that the angle F H C is equal to the angle G H C.

One hint, however, it is but fair to give, in order to put the learner on his guard against a mistake very commonly made in the demonstration of the Twelfth Proposition. It is often said the base C F is equal to the base C G, because they are sides of an equilateral triangle. But if the learner will look back to the construction, he will *not* find that C F G was made an equilateral triangle; but he *will* find that E F G is a circle, of which C is the centre, and therefore (as already often pointed out)

[1] In the figure here given the construction required in order to get the point H is given in dotted lines. It is hoped that the learner, by referring to the Tenth Proposition, will understand the construction without further explanation.

he may be sure that in the demonstration he will have to say 'F C is equal to C G because they are radii of the same circle.' With this one precautionary hint, it is hoped that few will have any difficulty in writing out the five steps of the reasoning in the demonstration.[1]

[In the hints accompanying the last four problems, the connection between the construction and demonstration is, some may think, to a needless degree pressed on the learner's notice.

The writer thinks it may be useful to some to do so. Beginners do not, generally, link together the construction and the demonstration. When they have learned to do so, their power of grappling with a proposition of Euclid is greatly and *most legitimately* increased.]

It remains only to write out the Twelfth Proposition formally, as it is given in Euclid.

THE TWELFTH PROPOSITION WRITTEN OUT.

PROPOSITION XII. PROBLEM.

General Enunciation.

A straight line of unlimited length being given, and also a point without it:—

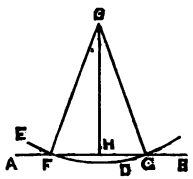
It is required from the given point to draw a straight line perpendicular to the given straight line.

Particular Enunciation.

Let A B be the given straight line (see fig. p. 118), which may be produced to any length both ways; and let C be the given point without it:—

[1] If any learner still looks for guiding hints, he can take them from those given for the Eleventh Proposition.

It is required to draw from the point C a straight line perpendicular to A B.



Construction.

1. On the other side of A B take any point D.

2. From the centre C, at the distance C D, describe a circle E F G, cutting A B in the points F and G.

3. Bisect F G in the point H (Prop. X.).

4. Join C H.

Then will C H be perpendicular to A B.

5. For the demonstration, join C F and C G.

Demonstration.

1. In the two triangles F C H and G C H, the side F H is equal to the side H G, because it was made so.

2. And C H is common to both triangles.

3. Wherefore the two sides F H, H C of the triangle F H C are equal to the two sides G H, H C of the triangle G H C.

4. And the base C F is equal to the base C G, because they are radii of the same circle.

5. Therefore by the Eighth Proposition the included angle F H C is equal to the included angle G H C.

And these are adjacent angles; but when one straight line falling on another straight line makes the adjacent angles equal, each of them is a right angle, and the straight line which stands on the other is called a perpendicular to it (Def. 10).

Wherefore C H is perpendicular to A B, and it is drawn from the point C. Q.E.F.

CONCLUSION.

The writer has now completed the task which he undertook, viz., to give a familiar explanation of the first twelve propositions of Euclid. His object all through has been to teach beginners how to learn Euclid. The aim of his explanations and observations has been throughout to prevent Euclid being, to the reader, merely a book of words, and to make it to him a book of sound and solid meaning.

He hopes that the student will go through the propositions that follow successfully and with benefit to himself, by following the method of learning them here pointed out. Namely :—

First. By fixing in his mind what is given, and what has to be done or to be proved.

Secondly. By observing the construction by which Euclid professes to effect what he wants to effect; *building up* his figure according to Euclid's directions.

Thirdly. By following out the demonstration in order to discover if Euclid therein shows that the construction has effected what he *says* it has.

He would even recommend the learner to do more than this. He would advise him, in his further progress through the First Book, to carry on the method of reading Euclid suggested by the writer's treatment of the last four propositions.

When the student has mastered the enunciation and construction, let him close his book and try whether he can make out for himself the proof that the construction has effected what was required.

If he cannot succeed without help, then let him note all the references, and observe the order in which they succeed each other in Euclid's demonstration. These references

will afford suggestive hints analogous to those given in the preceding pages, in order to encourage and help the learner to make out for himself the demonstrations of the Ninth, Tenth, Eleventh, and Twelfth Propositions, before reading over the demonstrations as given in Euclid.

When Euclid is thus read it becomes much more interesting. It has all the entertainment of solving riddles. Riddles, indeed, of a very high order are Euclid's problems and theorems.

This method has also the effect of fixing the propositions in the learner's mind far more permanently than if he merely reads through Euclid's demonstration, so as to follow his meaning, but without exerting independent thought.

Still, the learner must be cautioned while reading Euclid in this way not to hurry on too much, but to compare his demonstration with Euclid's, and to write it out in the best form with attention and care.

It is related in Sir David Brewster's 'Life of Sir Isaac Newton,' that, having purchased a copy of Euclid, soon after entering at Trinity College, he examined the problems, and found the truths which they enunciated so self-evident, that he expressed his astonishment that any person should have taken the trouble of writing a demonstration of them. He therefore threw aside Euclid as a 'trifling book,' and set himself to the study of Descartes' Geometry. The neglect which he had shown of the elementary truths of geometry he afterwards regarded as a mistake in his mathematical studies, and he expressed to Dr. Pemberton 'his regret that he had applied himself to the works of Descartes and other algebraical writers before he had considered the elements of Euclid with that *attention* that so excellent a writer deserved.'—Sir Isaac Newton's words as given in Pemberton's view of his philosophy.

Explanation of the Frontispiece.

Obs. The pegs from which the chains hang may be considered to represent the axioms on which the propositions depend.

1. The First Proposition is the first link of the chain.

2. The Second Proposition, depending on the First, is the second link.

3. The Third Proposition, depending directly on the Second, and through the Second on the First, is the next link.

4. These three propositions are represented by smaller links, they being propositions subsidiary to those of the main chain of propositions, which starts from the Fourth Proposition as from a fresh beginning.

5. Upon the Fourth hangs the Fifth Proposition; therefore the link (5) in the main chain is suspended from (4).

6. Upon the Fourth hangs also the Sixth Proposition; but no proposition among the first Twelve depends upon the Sixth, and so no link hangs from that numbered (6).

Obs. In the construction of the Fifth and Sixth Propositions it is required from the greater of two straight lines to cut off a part equal to the less (Prop. III.). Therefore a second and special link connects those numbered (5) and (6) with the one numbered (3).

7. The Seventh Proposition depends on the Fifth; therefore link (7) is hung on (5).

8. On the Seventh Proposition depends the Eighth; therefore link (8) is hung on (7).

9. The Eighth Proposition enables us to solve the Ninth; therefore link (9) hangs from (8).

10. For the construction of the Tenth Proposition the Ninth is required; therefore link (10) is hung from (9); but as it is demonstrated by the Fourth, a special link in the chain connects (4) and (10).

11. The Eleventh Proposition depends on the Eighth, therefore link (11) is hung from link (8); but as the construction requires that from the greater of two given straight lines a part be cut off equal to the less (Prop. III.), a special link connects those marked (11) and (3).

12. In the construction of the Twelfth, it is necessary to bisect a straight line; therefore, in the chain (12) is linked to (10). It is demonstrated by the Eighth; therefore a special link connects those marked (8) and (12).

QUESTIONS ON THE FOREGOING TREATISE.

To those learners who find that the explanations and questions are more than they require for a thorough understanding of Euclid's reasoning, the writer has but to repeat what he said in the preface to the first edition. Let learners do with these explanations and questions as they did with their swimming-belts when learning to swim. As soon as each one found he could swim alone, he flung away his swimming-belt, and dashed off into deep waters. So let him do with these explanations and questions. When he feels that he has learned how to learn Euclid, let him throw these helps aside, and take to his Euclid. The glorious six books of Euclid will afford him plenty of deep water to revel in.

INTRODUCTORY.

1. How long is it since Euclid wrote his treatise on geometry?

2. In what language did he write it?

3. And where?

4. What are the parts of Euclid commonly read in schools in England?

5. Distinguish between plane and solid geometry.

6. Which books of Euclid treat of plane geometry, and which of solid geometry?

On Points, Lines, Surfaces.

7. How many dimensions must a body have, which can be held in the hand?

8. What are they called?

9. Suppose the body is a wooden brick, when its thickness has been taken away, what two dimensions yet remain?

10. And what name does Euclid give to that which has the two remaining dimensions after the thickness has been taken away?

11. Now suppose the breadth to be also taken away, what name does Euclid give to that which has the single remaining dimension of length?

12. Suppose now the remaining dimension of length taken away, what name does Euclid give to that corner dot (the remains of the wooden brick), after its thickness, breadth, and length have been successively taken away?

13. Write Euclid's definition of a point (Def. 1).

14. It has been objected to Euclid's definition that it only tells what the point *has not*; what word has it been proposed to add to the definition of a point in order to tell something that it has?

15. If Euclid's point was to move along, what would it generate?

16. Write Euclid's definition of a line (Def. 2).

17. What are the extremities of a line?

18. What is the intersection of two lines?

19. Suppose the point in moving never swerves one way or another, but lies 'evenly' between its extreme points, what name does Euclid give to the line thus generated?

20. How may a surface be conceived to be generated from a line?

21. What are the extremities of a surface?

22. What method do the polishers of marble slabs use to ascertain if the surface is flat?

23. Write the definition of a flat surface (Def. 7).

24. How may a solid be conceived to be generated from a surface?

25. Suppose you have before you a flat surface (*see* Def. 7), on which to draw lines, and to place points, what must you conceive in reference to the lines you draw, and the dots you place on it to denote points, in order that they may become mathematical lines and points?

Read over the questions 7 to 12; also the questions 13, 15, 20, and 24, and answer the following questions:—

26. What distinction do you notice between the method of considering the early definitions of Euclid suggested in the questions 7 to 12, and the method suggested in the questions 13, 15, 20, and 24?

27. What is the name given to the first method of considering the definitions, from two Greek words, meaning to take to pieces? (*See note.*)

28. What is the name given to the second method of considering them, from two Greek words meaning 'to put together'? (*See note.*)

On Angles.

29. Open your compasses. By what name does Euclid express the amount of opening between the legs?

30. Write the definition of an angle (Def. 9).

31. What may be learned by forming angles of different magnitude by opening wider or drawing closer together the legs of a compass?

32. By what word does Horace describe the 'corner' of his farm where he hoped to sit in his old age and enjoy himself?

33. Describe a form of compasses made for smiths in order to keep the legs at any desired distance from each other.

34. Exhibit and describe a mode of indicating an angle derived from compasses of this make.

35. What must a learner guard against imagining from this mode of indicating an angle?

36. If two straight lines only meet at a point, how does Euclid indicate the angle between them?

37. If more than two straight lines meet at a point, how does Euclid indicate the angles between them?

Draw the first figure on page 8, and answer the following questions:—

38. If to the angle B A C the angle C A D be added, what angle is made?

39. If from the angle B A D the angle B A C be taken, what angle remains?

40. Describe a droll and unexpected mistake sometimes made in answering question 39.

Draw the second figure given on page 8, and answer the following seven questions:—

41. If to the angle A B D the angle D B C be added, what angle is made?

42. If from the angle A B L the angle K B A be taken, what angle remains?

43. What angle is made by adding the angle L B M to the angle M B C?

44. What angle remains if the angle A B O be taken from the angle A B F?

45. Which is greater, the angle B A E or M A N? If one is greater than the other, by how much is it greater?

46. Which is the greater of the angles D B C, or E B F, or O B G?

47. Place the angles B A D, C A B, and N A M in order of magnitude.

48. If a ruler standing on its end is brought to the edge of your desk, and makes the angles, on either side of it, with the edge of the desk, equal to each other, what are these adjacent angles called?

49. If one of the angles is greater than the other, what is the greater angle called? what is the smaller angle called?

50. Write the definition of a right angle (Def. 10), also of an obtuse angle (Def. 11), also of an acute angle (Def. 12).

On a Circle.

51. What is meant by the boundary of anything?

52. What name does Euclid give to that which is contained by one or more boundaries?

53. Define a circle (Def. 15).

54. What is the centre of a circle (Def. 16)?

55. What is the diameter of a circle (Def. 17)? what is the radius of a circle?

On Rectilineal Figures.

56. What are rectilineal figures?

57. What are the three classes into which Euclid divides rectilineal figures?

58. What are the three classes into which Euclid divides triangles according to the length of their sides?

59. What are the three classes into which Euclid divides triangles according to the magnitude of their angles?

Draw any triangle A B C, and answer the following questions:—

60. If BC be taken as the base of the triangle A B C, which is the vertex? which is the angle at the vertex? which are the two sides of the triangle? which are the two angles at the base? which sides are opposite each of the angles at the base? what is opposite to the vertical angle?

61. Answer the same questions when A B is taken for the base.

62. Also when A C is the base.

63. What are the three aspects under which a triangle may be viewed?

64. What is meant when it is said of two triangles that they are equal in every respect?

65. Define a square (Def. 30).

On the Postulates.

66. What is the meaning of the word 'postulate'? whence is it derived?

67. What are the three postulates?

68. What instruments do the postulates imply may be used, and for what purpose?

69. Is there any other meaning to the postulates?

On the Axioms.

70. What is the meaning of the word 'axiom'? Whence is it derived?

71. What is the first axiom?

72. Is the following reasoning right or wrong? If wrong, correct it:—A is equal to B, and C is also equal to B, therefore A, B, C, are al equal to one another (axiom 1).

73. Write the 2nd, 3rd, 4th, 5th, 6th, and 7th axioms.

74. What do you observe in reference to the 6th and 7th axioms? (See note.)

75. Write the 8th axiom.

76. Illustrate its meaning by supposing you are casting bullets in a mould.

77. Write the 9th, 10th, and 11th axioms.

QUESTIONS ON THE PROPOSITIONS GENERALLY.

1. What is there in every proposition to start from? What is it called? Whence is the name derived?

2. If starting from this you are required to *do* something, what is the proposition called? Give an instance.

3. If you are required to *prove* something, what is the proposition called? Give an instance.

4. What is the heading of a proposition called?

5. What is it very important to understand, and distinguish, on taking in hand every proposition?

6. In order to do this, how should the enunciation be written?

7. Distinguish between the words *General* and *Particular* enunciation?

8. What is that which comes after the enunciation in most propositions?

9. What is the final part of every proposition?

10. What is to be learned from the story of the post-boy?

DISCUSSION OF PROPOSITION I.

1. Write the general enunciation in two sentences; in the first stating what is given, and in the second stating what has to be done?

2. Draw the given straight line A B, and write the particular enunciation, as above, in two sentences.

Construction.

3. Write the first step of the construction, and do it.

4. Write the second step, and do it.

5. Write the third step, and do it.

6. Write the final clause of the construction, in which Euclid states what it will be found that the construction has done.

Demonstration.

7. Prove that A C is equal to A B.

8. Prove that C B is equal to A B.

9. What straight line have A C and B C been each proved equal to?

10. What conclusion do you draw by axiom 1 from A C and B C being each equal to the same straight line?

11. Is the following right or wrong? if wrong, correct it. 'Therefore A B, B C, C A are all equal to one another, axiom 1.'

12. When you have shown that each of the straight lines A B, B C, C A is equal to each of the other two, what can you then say?

13. Write the final clause of the demonstration.

14. What is to be learned from the story of the man that went by the train?

15. In the triangle A B C which is the London and which the Windsor man? which is the man that went by the train?

16. Write out the first proposition without guidance, as it is done in Euclid.

DISCUSSION OF PROPOSITION II.

1. Write the general enunciation in two sentences; the first stating what is given; the second what has to be done.

2. Draw the given straight line B C, and the point A, and write the particular enunciation in two sentences, as above.

3. What must you carefully avoid doing, in drawing the figures (or diagrams) of all the propositions of Euclid?

Construction.

4. Write the first step of the construction, and do it.

5. By what postulate can you do this?

6. Write the second step, and do it.

7 The second step links the second proposition to what preceding proposition?

8. In order to draw the equilateral triangle on A B what must be the centres, and what the radii of the intersecting circles?

9. Instead of drawing the whole circles, how would a draughtsman get the vertex of the equilateral triangle?

10. Having drawn the equilateral triangle, write the third step of the construction, and do it.

11. How must you proceed in order to produce straight lines correctly?

12. Having produced the straight lines, write the next step of the construction and do it.

13. In like manner write, and do the last step of the construction.

14. Write the last clause which Euclid adds on to the construction.

Demonstration.

15. Prove that B C is equal to B G.

16, Prove that D G is equal to D L.

17. How do you know that D A is equal to D B?

18. Prove that A L is equal to B G.

19. What was B G proved equal to?

20. What two straight lines have you shown to be each equal to B G?

21. What follows by axiom 1?

22. Of the three straight lines A L, B C, and B G, which is the London and which the Windsor man, and which is the man that went by the train?

23. Name a remark given in the text which is a great help in writing out the demonstration of every proposition?

24. Illustrate the foregoing remark, by reference to the second proposition.

25. Now write out Proposition II. without guidance.

On Varieties in the Figure of the Second Proposition.

*** Answer the next five questions without using letters to indicate the points, or lines referred to.

26. What points do you first join?

27. On what line, as base, is the equilateral triangle described?

28. What are the straight lines which are to be produced?

29. What is the centre, and what the radius, of the first circle which is to be described?

30. What is the centre, and what the radius of the second circle to be described?

31. Besides the straight line which you have drawn from A equal to B C (fig. p. 33), draw a second straight line by describing the equilateral triangle on the other side of the line A B.

32. Draw a third straight line from A equal to B C, by joining the given point A with the other end C of the line C B, and describing the equilateral triangle on the upper side of A C.

33. Draw a fourth straight line, by describing the equilateral triangle on the other side of A C.

34. Let the given point A be placed in the straight line B C and from A draw four straight lines, first in separate figures, and secondly in the same figure, each equal to B C.

35. As the distance A B diminishes, what do you observe with regard to the length of sides of the equilateral triangle to be described on A B?

36. If A coincides with B what do the sides of the equilateral triangle become? and what do the two circles become?

37. In this case what is the whole construction required, and what is the number of the solutions of the problem, which the construction gives?

DISCUSSION OF PROPOSITION III.

1. What makes the point of the third proposition to be often missed?

2. Write out the general enunciation in two sentences, stating in the first what is given, and in the second what has to be done.

3. Draw the two given straight lines, and calling the greater of the two A B, and the other C L, write out the particular enunciation in two sentences as above.

Construction.

4. What does Euclid give as the first step of his construction in Proposition 3?

5. In order to do this first step, what proposition is needed?

6. Now, in order, by Proposition 2, from the point A, to draw a straight line equal to C L, what two points must you first join? (Join them lightly in pencil.)

7. On what straight line must you describe an equilateral triangle?

8. What proposition teaches you how to do this? (Do it in pencil.)

9. Calling the vertex of this equilateral triangle H, say what straight lines you have to produce. (Produce them in pencil to M and K.)

10. What is the centre, and what is the radius of the first circle, now to be described? (Describe it in pencil, and put the letter N where this circle intersects the straight line H M.)

11. What is the centre, and what is the radius of the second circle now to be described? (Describe it in pencil, and put the letter D where this second circle intersects the straight line H K.)

12. What straight line have you thus drawn from A equal to C L? (Ink that line over.)

13. What axiom do you use to prove that the straight line you have named in answer to question 12, is equal to C L?

14. What are the straight lines corresponding to the London and the Windsor man in this proposition?

15. What is the straight line corresponding to the man that went by the train?

16. When the straight line that you have 'inked in' is dry, efface all the pencil lines of the construction, and say what three straight lines you have remaining.

17. Of these what is A B?

18. What is C L?

19. What is the straight line A D equal to, and from the extremity of what straight line has it been drawn?

20. What is the single step of construction required for the new part of proposition 3? (Do it.)

21. Write the last clause of the construction, in which it is said what straight line will be equal to C L.

Demonstration.

22. Prove that A E is equal to A D.

23. What was A D made equal to? by what proposition?

24. Prove that A E is equal to C L.

25. Write the last clause of the demonstration, stating what straight line you have cut off equal to C L.

26. State what straight line in this proposition represents the man that went by the train; and what straight lines represent the Windsor man and the London man.

27. Write out proposition 3 as it is done in Euclid, without guidance.

28. If both the given straight lines are drawn from the same point, what is the whole construction required in order from the greater to cut off a part equal to the less.

DISCUSSION OF PROPOSITION IV.

1. What is the fourth proposition called?

2. The first three propositions are problems; what is the fourth?

General Enunciation.

3. Write what is given in the fourth Proposition.

4. Write in three sentences what has to be proved.

5. How may the equality of the two triangles in respect to sides, angles, and area be expressed in one sentence?

6. In drawing two triangles A B C and D E F which shall fulfil the three given conditions, how may D E be made equal to A B?

7. How may the angle E D F be made equal to B A C?

8. How may D F be made equal to A C?

9. How do you reconcile this use of instruments with the rule of not using the compasses, &c., to measure with?

10. What points have to be joined, in order to complete each of the triangles A B C and D E F?

Particular Enunciation.

11. Write in three sentences what is given in reference to the two triangles A B C and D E F.

12. Next write in three sentences what you have to prove.

13. By what axiom is the base B C to be proved equal to the base E F?

14. What must we therefore, by our reasoning, try to show?

15. In order to show that B C and E F may be made to fill the same space, where are you to conceive the triangle A B C to be laid? and where are you directed to place A, and where A B.

16. Might you conceive the triangle A B C to be placed as here directed whatever its form might be?

17. Show by a figure, or drawing, that you might lay your 'set square' on a book in the way Euclid directs you to lay the triangle A B C on the triangle D E F.

18. Why would it be wrong to say put A on D, and B on E, and C on F?

19. A lying on D, and A B lying along D E as far as it goes, where will B come exactly? and why?

If you cannot answer question 19, answer the following three questions:—

20. Suppose you have two walking-sticks, one of them *two inches longer* than the other, if the handles of the sticks are put together, and one stick is made to lie along the other as far as it goes, will the points of the walking-sticks come together? why not?

21. If the sticks are of *equal* length and are placed as stated in question 20, will the points come together? why?

22. Now suppose A B and D E to be the two sticks; if A is placed on D, and A B lies along D E, will B come on E? why?

23. Once more take up your set square and some book, put one of the acute angles of the set square on the corner of the book, and a side of the set square along the side of the book, does the other side of the set square come along the top edge of the book? why not?

24. But if you turn round the set square and put the right angle of it on the corner of the book, and one side of the set square along the side edge of the book does the other side of the set square now lie along the top edge of the book? why?

25. Now turn to the triangles A B C and D E F (fig. p. 49); A is placed on D and A B lies along D E, will A C lie along D F? why?

26. What was the way a boy once said this proposition and what was the correction?

27. What was the future post captain's question?

28. What was the master's explanation of his difficulty?

29. Having shown that A C will lie on D F as far as it goes where will C come? and why?

30. If it was not given that A B was equal to D E, what could you *not* prove?

31. If the angle B A C was not given equal to the angle E D F, what could you *not* prove?

32. If the angle B A C were given equal to E D F, but A C was not given equal to D F, what could you *not* prove?

33. What follows from questions 30, 31, 32?

34. Why, in the figure on page 46, are strokes drawn along the triangle A B C?

35. Why are the second set of letters put at the angular points of the triangle D E F?

36. To what straight lines are you now to confine your attention?

37. B and C are the beginning and end of one of the bases, E and F the beginning and end of the other base; where has it been shown that B and C must come if the triangle A B C be laid on D E F as directed?

38. Is it possible for the beginning and end of each base thus to come together, and the two bases to separate at all between the beginning and end? why not?

39. To illustrate. If you have a couple of pens, and bring the feather ends together and the quill ends together, the pens separate as

in the figure page 51; why? What axiom teaches that they could not separate if they were straight?

40. You have now shown with reference to B C and E F, first that they begin together; secondly, that they end together; thirdly, that they keep together from beginning to end; what follows from this?

41. What says the 8th axiom?

42. What then can you now say of B C and E F?

43. Why is it only necessary to show that the two bases occupy the same space in *length*? (See note.)

44. What are the angles of the triangles A B C and D E F, to which the equal sides A C and D F are opposite?

45. In order to prove that these angles (viz., A B C and D E F) are equal, what must you show in reference to them?

46. Where is A B lying now? How do you know?

47. Where is B C lying now? How do you know?

48. If now, you draw a curved rim to indicate the angle A B C, and another to indicate the angle D E F, you find that you draw both the curved rims through what space?

49. What then can you now say of the two angles A B C and D E F?

50. What says the 8th axiom?

51. What then can you say of the angles A B C and D E F, by the 8th axiom?

52. Again in the triangles A B C and D E F, what are the angles to which the equal sides A B and D E are opposite?

53. Where is A C now lying? How do you know?

54. Where is C B now lying? How do you know?

55. Then what is the space, or opening, filled by each of the angles A C B and D F E?

56. What says the 8th axiom?

57. What then can you now say of the angles A C B and D F E by axiom 8?

58. What remains to be proved in reference to the triangles A B C and D E F?

59. Where do you know that A B is lying? How do you know this?

60. Where is A C lying? How do you know?

61. Where is B C lying? How do you know?

62. If now you put the point of your finger on the space filled by the triangles A B C and D E F, into what space do you put your finger, both times?

63. And what does the 8th axiom say?

64. What then, can you say of the included areas or surfaces, by axiom 8?

65. Why, in showing that the areas fill the same space, need you only think of the boundary lines lying upon each other? (See note.)

66. What have you now shown with reference to the two triangles A B C and D E F?

67. Now write out the fourth proposition, without hints or guidance, as it is done in Euclid.

68. What is the way to find which of the angles at each base form the pair of angles equal to each other?

69. Instead of using the words 'apply one triangle to the other,' or 'lay one on the other,' which might suggest the idea of thickness, what words has it been proposed to use, in order to make the learner feel that the triangles when brought together—as Euclid directs they shall be—will occupy the same space.

EXERCISES ON THE FOURTH PROPOSITION.

1. Having once for all proved the fourth proposition, what will you be able henceforth to say of any two triangles which have two sides of the one equal to two sides of the other, and have likewise the included angles equal?

EXERCISE 1.

2. Draw the given triangle A B C (see page 61), and write down what is given (in the enunciation) in reference to the two sides A B and A C, and then what you are allowed to take for granted in reference to the angle B A C.

3. Write in a second sentence what you are required to prove.

4. Which are the two triangles to be compared?

5. What are the bases of these triangles? add how you know which of the three sides in each triangle is to be taken as the base?

6. Knowing the bases, say which are the sides of the triangles A B D and A C D?

7. It being given (see quest. 2), that the straight line A B is equal to A C, write as step 1 of the demonstration, that it is so, giving the reason why you are able to say that it is.

8. If on looking at the triangles A B D and A C D you see a reason for saying that a second side of one of the triangles is equal to a second side of the other, put down as step 2 that it is so, and give the reason why.

9. As step 3, club together the two sides of one triangle, which you have named (in steps 1 and 2), and say they are equal to the two sides you have named of the other triangle, each to each.

10. Look now at the two sides that you have thus named of the triangle A B D, namely B A, A D; and say what is the angle contained (or included) by these sides.

11. In like manner look at the two sides you have named of the triangle A C D, viz., C A, A D, and name the angle contained by these sides.

12. Now look back at the enunciation, and observe what is there said of the two angles you have named in answer to questions 10 and 11. And write as step 4 of the reasoning, that these included angles are equal, and add the reason why you are able to say that they are equal.

13. Having shown that two sides of one of these triangles are equal to two sides of the other each to each; and that the included angles are equal, write as step 5 what you can say of the two triangles A B D and A C D, by the fourth proposition.

14. Once more look back to the enunciation and observe in what respect you were required to prove the triangles A B D and A C D to be equal, and fill up the blanks in the following sentence:—'Wherefore the . . . . . is equal to the . . . . . which was to be proved.'

15. Suppose that in this exercise, instead of being required to prove the bases equal, you had been told to prove that A D was at right angles to B C, how would you fill up the blanks after the word, 'Wherefore,' in question 14?

16. What definition would you then have to refer to, in order to prove that A D B and A D C are right angles?

17. Now write out the enunciation and demonstration of Exercise 1, without guidance.

EXERCISE 2.

1. Draw the straight line A B (see fig. page 65).
2. Where is it given that the point C is?
3. How are you allowed to consider that C E is drawn?
4. Where is D taken, in the straight line C E?
5. What points do you join? Join them.
6. What is required to be proved?
7. Which are the two triangles to be compared?
8. Which are the bases; how do you know?
9. Which are the two sides of the one triangle?
10. And which are the two sides of the other?
11. Which of these sides are given equal? (see enunciation).
12. Which side is common to both triangles?
13. Which is the angle contained by the two sides named of the first triangle, and also of the second triangle?
14. How do you know that these angles are equal?

Now write out the five steps of the proof, as follows:—

15. Step 1.—Say that one side in each triangle is equal, and give the reason why.

16. Step 2.—Name the side which is common to both triangles?

17. Step 3.—Club together the sides of each triangle thus named as equal, each to each.

18. Step 4.—Say if the included angles are equal, and give the reason why.

19. Step 5.—State what follows by proposition 4, and specify the particular respect in which you were told in the enunciation to prove the triangles equal.

20. Now write the five steps of the demonstration without guidance.

EXERCISE 3.

1. Draw the given triangle A B C (see fig. page 66), and write where you are allowed to consider the point E to be taken.

2. What points are to be joined? (join them).

3. What line is to be produced? (produce it).

4. What is the produced line to be made equal to? (make it so by prop. 3).

5. What points are then to be joined? (join them).

6. What is required to be proved?

7. What is allowed to be taken for granted in doing this Exercise which will be afterwards proved in the 15th proposition?

8. Which are the two triangles you are about to compare by the 4th proposition?

9. Before writing out the five steps of the proof, state the first rule given in the text for finding out which of the three sides of two triangles to be compared, ought to be taken for the base?

10. Name the base which is pointed out by rule 1?

11. State the second rule given, for finding out the base?

12. Name the base here pointed out by rule 2?

13. State the third rule given for finding out the base?

14. Name the base here pointed out by rule 3.

15. Knowing which the base is, name the two sides in each triangle to be compared?

Now write out the five steps of the proof in the following order:—

16. Step 1.—Name one side in one of the triangles to be compared, which is equal to one side of the other triangle, and give the reason why you are able to say that they are equal.

17. Step 2.—Do the same for a second side in each of the triangles to be compared.

18. Step 3.—Club together the two sides you have named in steps 1 and 2, of one of the triangles, and say they are equal to the two sides you have named in steps 1 and 2, of the second triangle, each to each.

19. Step 4.—Note the angles contained by the sides of the triangles named in step 3. If you are allowed (see question 7) to consider these angles equal, say that they are so, and give the reason why you are allowed to consider them equal.

20. Step 5.—State, lastly, the conclusion you draw from the above conditions (or premises) by prop. 4.

Now look back to the enunciation, and notice what are the angles required to be proved equal, and continue your answer to question 20, thus:—Wherefore the angle B A E is equal to the angle (name it), these being the angles to which the equal sides (name them) are opposite. Also the angle A B E is equal to the angle (name it), these being the angles to which the equal sides (name them) are opposite.

21. Now on a fresh page draw the given triangle A B C, write where E is allowed to be taken, name and draw all the straight lines required

to be drawn in order to complete the figure; specify the angles to be proved equal: and write out the proof that they are equal without further guidance.

EXERCISE 4.

1. Before doing the 4th exercise, write the definition of a square (Def. 30). Also write out axiom 11.

2. What figures are you allowed to assume to be described on the straight lines A B and B C (see fig. page 68)?

3. In what proposition will you be taught hereafter to describe a square on a given straight line?

4. Draw now the triangle A B C, and on the sides A B, B C, you may describe the squares A B D E, and B C G F, with the help of your set-square and compasses, inasmuch as it is given in the enunciation that they are squares.

5. What points are to be joined?

6. What lines have to be proved equal?

7. What are the triangles that have to be compared?

8. Which straight lines are to be taken as the bases of these triangles? How do you know?

9. Name the two sides of each triangle.

10. Is D B equal to A B? Why?

11. Is B C equal to B F? Why?

12. What is the angle included between the two sides of the triangle D B C? Name also the angle contained by the two sides of the triangle A B F.

13. Of what two angles is the angle D B C made up?

14. Of what two angles is the angle A B F made up?

15. Is the angle D B A equal to the angle C B F? Give the reason why.

16. What angle is common to both of the angles D B C and A B F?

17. What do you conclude by axiom 2, in reference to the angles A B F and D B C?

Now write down the five steps of the demonstration required to prove that A F is equal to D C in the following way:—

18. 1st Step.—Show that one side in each triangle to be compared is equal.

19. 2nd Step.—Show the same for a second side in each triangle.

20. 3rd Step.—Club together, as equal, each to each, the two sides named in steps 1 and 2 as sides of each of the triangles to be compared.

21. 4th Step.—In order to prove that the included angles are equal, first show that two angles, which are parts of the included angles, are equal. Secondly, name a common angle which when added to the above-named equal parts, makes them the angles included between the two sides of each of the triangles to be compared; name also the axiom by which you are able to say that they are equal.

22. 5th Step.—Complete the demonstration, specifying that the bases are equal.

23. Now write out the 4th exercise without guidance.

EXERCISE 5.

1. What is given in Exercise 5, in reference to the two circles E C G and D B F? (See fig. page 70.) Describe the circles.

2. The radii A B C and A D E being drawn, what points are to be joined? (join them.)

3. What straight lines, according to the enunciation, are to be proved equal?

4. And what angles are to be proved equal?

5. Which are the two triangles to be compared?

6. What are their bases? How do you know?

7. And therefore what are their sides?

8. Show that one side of one of the triangles is equal to one side of the other?

9. Show the same for a second side, in each triangle?

10. What is the angle contained by the two sides, above-named, of the triangle A B E?

11. What is the angle contained by the two sides above-named, of the triangle A D C?

12. Are these angles equal? Why?

13. Now that you have shown that two sides of the triangle A D C are equal to two sides of A B E, and that the included angles are equal, or common to both triangles, what can you say of the bases D C and B E, and of the remaining angles to which the equal sides are opposite?

14. What are the angles opposite the equal sides A C, A E?

15. What are the angles opposite the equal sides A B, A D?

16. Now write the five steps of the demonstration without guidance.

EXERCISE 6.

1. What further construction beyond that of the 5th exercise is required for the 6th exercise? (See fig. page 71, and draw the required line.)

2. What is required to be proved in the 6th exercise?

3. Which are the triangles to be compared?

4. Of the three angles of the triangles A C D and B C D, which angle is common to both the triangles A C D and B C D?

5. Also of the three angles of the triangles A E B and D E B, which is the angle which the triangles A E B and D E B have in common?

6. Which, then, of the angles of the triangles B C D and B D E do we know to be equal?

7. What straight line is then (by rule 2, page 67), the base of each of the triangles B C D and B D E?

8. Having ascertained the base, say what are the two sides of each of the triangles, B D C and B D E?

9. Is one of the sides in each of these triangles equal (viz. D C and B E)? How do you know?

10. Prove that the other side D E is equal to B C (by axiom 3).

11. Club together the two sides of each of the triangles you are now comparing, which you have shown to be equal, each to each.

12. Name the included angles, and say how you know them to be equal.

13. What can you now say of the two triangles B C D and B E D by the 4th proposition ?

14. Which are the angles opposite to the equal sides C D and B E ?

15. Which are the angles opposite to the equal sides B C and D E?

16. Now write out the five steps of the demonstration that the triangles B C D and D E B are equal in every respect, specifying the remaining angles that are respectively equal.

Deduction from Exercises 5 and 6.

17. In the 5th exercise what angle was A B E proved equal to?

18. In the 6th exercise what angle was D B E proved equal to ?

19. What angles can you infer from questions 17 and 18 are equal to each other, by axiom 3 ?

EXERCISE 7.

Obs.—This last exercise cannot be proved by Prop. 4. You are to show why it cannot.

1. What is given in this exercise ? (See second fig. page 74.)
2. What has to be proved ?
3. What construction is needed?
4. Write out the five steps of the incorrect proof, which in the text it is stated was once given of this exercise ?
5. Show where the reasoning above given is incorrect, and why it is incorrect ?

DISCUSSION OF PROPOSITION V.

(See Fig. Page 77.)

General Enunciation.

1. What is given in Proposition 5 ?
2. What has to be proved ?
3. If the two equal sides are produced, what further has to be proved ?

Particular Enunciation.

4. Draw the given isosceles triangle A B C, and name the sides which are given equal.
5. Name the angles which have to be proved equal.
6. What are the sides to be produced ? (Produce them to D and E.)
7. What are the other angles which have to be proved equal ?

Construction.

8. Where is the point F taken?

9. What is A G made equal to? What points are to be joined?

Demonstration.

[Before you begin the demonstration, first, observe the angles A B G and A C F, and *fix them in your mind.* Then observe the angles C B G and B C F, and *fix them in your mind.* Then answer the following questions.]

10. If the angle C B G be taken from the angle A B G, what angle remains?

11. If the angle B C F be taken from the angle A C F, what angle remains?

12. If then the angles A B G and A C F were proved equal and also the angles C B G and B C F, what could you say of the angles A B C and A C B by axiom 3?

13. Of what triangle is the angle A B G an angle?

14. Of what triangle is the angle A C F an angle?

15. What then are the two triangles you will have to prove equal in every respect in order to show that the angle A B G is equal to the angle A C F?

16. Now turn back to Exercise 5 and comparing the fig. there given with the fig. of Proposition 5, say what are the triangles in Exercise 5 which correspond to the triangles A B G and A C F in Proposition 5.

17. Name the bases of the triangles A B G and ACF. Why are they so?

18. Knowing the bases, name the two sides of each of the triangles A B G and A C F.

19. Name the angle contained by the two sides of the triangle A B G.

20. Name also the angle contained by the two sides of the triangle A C F.

21. Are the angles named in your answers to questions 19 and 20 equal? Why?

22. Now, with these hints as helps, write out the 5 steps of the proof that the triangles A B G and A C F are equal in every respect.

23. Specify the bases of the triangles A B G and A C F thus proved equal.

24. Specify also the angles opposite the two sides A B and A C in the triangles A B G and A C F which are thus proved equal.

25. Also specify the angles opposite the two sides A F and A G in the same triangles, thus proved equal.

26. Now look back to question 12, and say what progress you have made towards demonstrating that the angle A B C is equal to the angle A C B.

27. Repeat once more what are the other pair of angles you have yet to prove equal, in order to be able to say, therefore by axiom 3 the angle A B C is equal to the angle A C B.

28. The angles you have named in answer to question 27, are angles of what triangles?

29. Your answer to question 28 points out the next pair of triangles which you will have to prove equal in every respect by Proposition 4; what are they?

30. Passing then to these triangles, viz., the triangles B C F and C B G, which of the three angles of these triangles do you know to be equal to each other?

31. From your answer to question 30, say which are the bases of the two triangles B C F and C B G?

32. Hence say which are the sides of each of the triangles B C F and C B G?

33. As you proved in Exercise 6, that B C was equal to D E (see fig. p. 72) so prove here that the side B F of the triangle B F C is equal to the side C G of triangle C G B.

34. Also say how you know that the side F C of the triangle F C B is equal to the side G B of the triangle G B C.

35. Now keeping in mind the proof given in Exercise 6 that the triangles D B E and B D C (see page 72) were equal in every respect, write the five steps of the demonstration that the triangles B C G and F B C are equal in every respect; and specify the angles in each of the triangles B C G and F B C that are equal, as being those to which the equal sides are opposite.

36. Now that you have proved that the angle A B G is equal to A C F (see question 27), also that the angle C B G is equal to the angle B C F (see question 35); prove now that the angle A B C is equal to the angle A C B.

37. Look now at the angles C B F and B C G. Do you know that they are equal? If you do, say when or how you proved them equal?

Look back at the enunciation. And if you feel that in the demonstration you *have proved* those two pairs of angles equal, which you stated, in the enunciation, had to be proved equal, write as the conclusion of the demonstration that they are equal, each to each, naming them.

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38. When you proved that the two triangles A B G, A C F were equal in every respect (see fig. page 77), you specified three respects in which the triangles were equal: what were they?

39. When you proved that the two triangles B C F and B C G were equal in every respect, you specified two respects in which they were equal: what were they?

40. What use is made in the subsequent reasoning of:—

i. The base B G being equal to the base C F?

ii. Of the angle A F C being equal to the angle A G B?

iii. Of the angle A B G being equal to the angle A C F, and of the angle C B G being equal to the angle B C F?

iv. Of the angle F B C being equal to the angle G C B?

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41. Now that you have gone through Proposition V. in this disjointed way, write it out, as it is done in Euclid, without guidance.

42. Euclid adds, as a corollary, or deduction, from the fifth proposition, that it can be shown that every equilateral triangle is also equiangular. Prove that it is so.

DISCUSSION OF PROPOSITION VI.

1. Write the general enunciation of the sixth proposition.

2. Write also the first part of the general enunciation of the fifth proposition, which speaks of the angles at the base being equal.

3. What relation do you observe between these two enunciations?

4. What is the name given to the sixth proposition in relation to the fifth?

5. Draw the given triangle A B C (see fig. page 83).

6. What angles are given equal to each other in the triangle A B C?

7. What sides have to be proved equal?

8. Can Euclid prove in a *direct* way that the side A B is equal to the side A C?

9. The *in*direct way in which Euclid proves that A B is equal to A C is by showing what?

10. Suppose an opponent, or inquirer, were to say, 'I deny that A B is equal to A C,' how does Euclid meet this denial?

11. What is the opponent obliged to reply to what Euclid says?

12. The opponent being obliged to admit that, if A B and A C are not equal, one of them must be greater than the other, what does Euclid say that he can, by the third proposition, cut off from the greater (say A B)?

13. Supposing this to be done; what points does Euclid join?

14. What are the two triangles which Euclid has now before him, which he can prove equal to each other in every respect, by the fourth proposition?

15. What are the bases of these two triangles? How do you know?

16. Now go through the five steps of the proof, that the triangles A B C and D B C are equal in every respect.

17. What is the particular respect in which, here, Euclid specifies that the triangles are equal?

18. Can they be equal in this respect?

19. Why not?

20. When a manifestly absurd conclusion is reached by an argument, it must be either because the hypothesis is—what? or the reasoning is—what? (See note.)

21. You have followed Euclid's reasoning. Is it unsound?

22. Then it must follow that the hypothesis is--what?

23. What was the hypothesis (or assumption) made by the opponent, and accepted by Euclid *for the sake of argument*?

24. If then it is false to say that A B and A C are *un*equal, what must they be?

25. What is the name given to this indirect proof?

26. Write out the sixth proposition as it is done in Euclid.

DISCUSSION OF PROPOSITIONS VII. AND VIII. TAKEN TOGETHER.

1. Write the general enunciation of the eighth proposition.

2. Write also the enunciation of the fourth proposition as follows:— It is given that two triangles have two sides of the one equal to two sides of the other, and have likewise the angles contained by those sides equal:—It is required to prove that their bases shall be equal.

3. Having written these enunciations say what relation you observe between the eighth proposition, and the fourth as it is given above.

4. Draw the two triangles given in Prop. 8, viz., A B C and D E F (see fig. page 93), and write the particular enunciation in two sentences; stating in the first what is given, and in the second stating what has to be proved.

5. Where, with the view of showing that the included angles B A C and E D F will fill the same space, does Euclid tell us to conceive one of the triangles (viz., A B C) to be placed? Where does he place B? Where does he make B C to lie as far as it goes?

6. Where does he prove that C will come? and why will it?

7. Why cannot he prove directly that, B C lying on EF, B A must lie on E D, and C A on F D, according to the method of proof which he used in the fourth proposition?

8. Not being able to prove directly that B A, A C must lie on E D, D F, he approaches the question indirectly by saying thus:—If B A, A C do not lie along E D, D F, they must take some other direction, in which case, he further says, either each of the vertices must lie . . . . or one of the vertices must lie . . . . or . . . . [complete these sentences.]

9. Draw three figures, representing the two triangles, applied one to the other; placing them in each of the three positions, one of which they must occupy, unless B A, A C lie along E D, D F. (See page 94.)

10. If, then, he can prove that the triangles cannot lie in any of the three positions which he names, where must B A, A C lie?

11. And if B A, A C lie along E D, D F, what can be said with respect to the angles B A C, and E D F? (Don't say that they are equal.)

12. And if the angles B A C and E D F fill the same space, what can be said of them by axiom 8?

13. *Suppose*, first, that each of the vertices falls outside the other triangle (see fig. page 95), what is the construction?

14. Look now at the triangle E D A, and from the *data* (i.e., what is given), say what kind of triangle E D A is.

15. Then what two angles of the triangle E D A are equal, by Proposition 5.

16. Recalling the illustration of the twin brothers, what angles of triangle E D A represent the twins?

17. Which angle is less than E D A, and therefore represents the younger brother (or D. White)? and what angle is greater than E A D, and therefore represents the elder brother (or Mr. Davis)?

18. Now prove that the angle F A D is greater than F D A.

19. But because F A is equal to F D, what follows in reference to the angles F A D and F D A?

20. What absurdity, then, has come from the supposition (made, remember, for the sake of argument), that each of the vertices falls outside the other triangle?

21. What, then, must we conclude with reference to the supposition that each of the vertices falls *outside* the other triangle? (Don't forget your answer to this question.)

22. Next, supposing (still for argument's sake) that one of the vertices falls inside the other triangle; what further construction is required? (See p. 96.)

23. What angles represent the twins now? Which angle represents the younger brother? and which angle represents the elder brother?

24. Prove now that the angle F A D is greater than the angle F D A.

25. But because F A is equal to F D, what follows in reference to the angles F A D and F D A?

26. What absurdity, then, has come from supposing that one of the vertices falls *inside* the other triangle?

27. What, then, must we conclude with reference to the supposition that one of the vertices falls within the other triangle? (Don't forget your answer to this question.)

28. What would follow from supposing that one of the vertices falls on the side of the other triangle? (See fig. page 97.)

29. What do we, therefore, conclude with reference to the supposition that one of the vertices falls *on the side* of the other triangle? (Remember this also.)

30. What have we shown in questions 21, 27, and 29?

31. Where, then, must the two straight lines B A, A C lie? (See fig. page 95.)

32. And what, therefore, is the space filled by each of the two angles B A C and E D F?

33. And what, therefore, is thus proved in respect of the angles B A C and E D F by axiom 8?

34. Reviewing Euclid's indirect proof—drawn out in the foregoing questions—that (if one of the given triangles be applied to the other as he directs) the two sides of one of the triangles can lie no where but on the two sides of the other; answer the three following questions:—

i. What use does he make in his reasoning of the given condition that the bases are equal?

ii. What use does he make of the given condition that the sides A B and D E are equal?

iii. What use does he make of the given condition that the sides A C, D F, are equal?

35. In dividing the seventh and eighth propositions (which have been here discussed together) into two propositions, what does Euclid make the enunciation of the seventh proposition? (See page 98.)

36. What is Euclid's statement, in the enunciation of Prop. VII., equivalent to saying?

37. Enunciate once more Prop. VIII.

38. When it has been shown by Euclid's method of applying one of the triangles to the other, that the base of each triangle occupies the same space, what have you got before you?

39. What, in that case, does the reasoning of the seventh proposition prove?

40. Write out the seventh proposition as it is done in Euclid.

41. Write out the eighth proposition as it is done in Euclid.

ON THE USE OF THE FOURTH AND EIGHTH PROPOSITIONS.

42. Having proved the fourth and the eighth propositions, what are these two propositions henceforth in your hands?

43. If it is given that in the triangle A B C, the side A B is equal to the side A C, and that the angle B A C is bisected by the straight line A D meeting the base B C in the point D; is it by the fourth or the eighth proposition that B D is proved equal to D C?

44. Write the five steps of the demonstration which proves that B D is equal to D C. (See question 43.)

45. If it is given that in the triangle A B C, the side A B is equal to the side A C, and that the base B C is bisected in D, then if A D be joined, is it by the fourth or the eighth proposition that the angle B A D is proved equal to C A D?

46. Write the five steps of the demonstration which proves that the angle B A D is equal to C A D?

DISCUSSION OF PROPOSITION IX.

Enunciation.

1. What is given in Prop. IX.?

2. What has to be done?

3. Draw the given angle (see fig. page 107).

4. Write the particular enunciation in two sentences. In the first say what is given; in the second, what has to be done.

Construction.

5. How do you get the point D in the straight line A B?

6. How do you get the point E in the straight line A C?

7. What is next done?

8. What is then described on the side of D E remote from A?

9. What points are then joined?

10. What do you now say of the angle B A C?

Demonstration.

11. Is it the fourth or the eighth proposition that is used in the demonstration? How do you know?

12. Which are the two triangles to be compared?

13. Which are the bases of the triangles to be compared? How do you know?

14. What, then, are the two sides in each triangle?

15. What are the angles contained by the two sides in each triangle?

16. Where must we look for an answer to the question, 'Is the side A E equal to A D?'

17. What side is common to both triangles?

18. Where must we look for an answer to the question, 'Is the base D F equal to the base E F?'

19. Write now the five steps of the demonstration that the angle D A F will be equal E A F.

20. Now write out the ninth proposition as it is given in Euclid.

21. Why is it said, Describe the equilateral triangle on the side of D E remote from A?

22. Draw fig. given on page 109 in which the equilateral triangle is described on the same side of the base D E as A is, and prove, using that figure, that by joining A F, the angle B A C will be bisected.

DISCUSSION OF PROPOSITION X.

Enunciation.

1. What is given in the tenth proposition?
2. What is required to be done?
3. What were we taught to do in the ninth proposition?
4. How is the tenth proposition linked to the ninth?
5. Draw the given straight line A B. (See fig. page 110.)
6. Write in two sentences, 1st, what is given, and, 2nd, what is required to be done.

Construction.

7. Write the first step, and do it.
8. What are you directed to do as the second step of the construction?
9. Now, in order to do this second step by Prop. IX., what is the first step of the construction required? What is the second step?
10. What points are to be joined?
11. On what line is the equilateral triangle described?
12. If you were not to describe the equilateral triangle on the side of its base remote from A, where would its vertex fall?
13. What points have to be joined in order to bisect the angle A C B?
14. Now efface all the construction required in order to bisect the angle A C B, as belonging to another proposition, except the bisecting line.
15. If the bisecting line does not intersect A B, what are you to do?
16. What do you say of the point D where the line bisecting the angle A C B cuts the given line A B?

Demonstration.

17. In order to prove that AB is bisected in the point D, what straight lines must you prove equal?
18. Which is it, the fourth or the eighth proposition, that you must use in order to prove that A D is equal to D B?
19. What triangles have you to prove equal?
20. Which are the bases of those triangles?
21. By what rule do you discover the bases?
22. What are the two sides and the included angle in each triangle?
23. How do you know that A C is equal to B C?
24. What is the side common to both triangles?
25. How do you know that the included angles are equal?
26. Write out the five steps required to prove that the base A D is equal to D B.
27. Write out the tenth proposition as it is done in Euclid.

DISCUSSION OF PROPOSITION XI.

Enunciation.

1. What is given in the eleventh proposition?
2. What is required to be done?
3. Draw the given straight line and take a point in it, and write out the particular enunciation in two sentences.

Construction.

4. What will Euclid want in order to solve the eleventh proposition?
5. How does he get what he wants? (See fig. page 113.)
6. What does he do on the straight line D E so obtained?
7. What points does he lastly join?
8. What does he say of the straight line F C?

Demonstration.

9. Before Euclid can say that F C is at right angles to A B, what angles must he show to be equal?
10. Is it the fourth or eighth proposition he will have to use in order to prove that those angles are equal?
11. Which are the triangles he will have to compare?
12. Which are their bases? How do you know that they are?
13. Which are the two sides of each triangle?
14. Which are the included angles?
15. Are the two sides equal, each to each? state why.
16. Are the bases equal? state why.
17. Write the five steps required to prove that the included angles are equal?
18. The included angles having been proved equal, what is Euclid able to say by the 10th definition?
19. Write out the eleventh proposition as is done in Euclid.

DISCUSSION OF PROPOSITION XII.

Enunciation.

1. What is given in Proposition XII.?
2. What is required to be done?
3. In what does the twelfth proposition differ from the eleventh?
4. Why are the words 'of unlimited length' introduced?
5. Now draw the straight line, take a point without it, and write out the particular enunciation in two sentences.

Construction.

6. How does Euclid proceed, in order to get two points in the straight line A B equally distant from C? (See fig. page 117.)

7. Whereabouts in the straight line F G does it appear to you the foot of the perpendicular from C will come?

8. Draw in pencil on the other side of F G all the construction required to get the point which will be the foot of the perpendicular drawn from C. Then efface all this construction, except the letter H.

9. What points do you now join to get the perpendicular required, and what do you say of the straight line C H?

10. What points have you to join for the demonstration?

Demonstration.

11. To prove that C H is perpendicular to A B what angles must be shown equal to each other?

12. What triangles have you to compare?

13. Which are the bases of those triangles? How do you know?

14. How do you know that the bases are equal?

15. What are the sides?

16. How do you know that F H is equal to H G?

17. Write the five steps of the demonstration required to show that the angle F H C is equal to G H C?

18. Having proved these angles equal, what can you say of C H?

19. Write out Proposition XII. as given in Euclid.

A SELECTION OF GEOMETRICAL EXERCISES REQUIRING ONLY THE KNOWLEDGE OF THE FIRST TWELVE PROPOSITIONS OF THE FIRST BOOK OF EUCLID.

*** *It is intended, and advised, that, throughout the following Exercises, the constructions shall be done carefully, according to Euclid's methods.*

1. Given a straight line:—It is required to produce it so as to make it altogether twice as long as the given straight line.

2. A straight line being given:—It is required, on that straight line as base, to describe an isosceles triangle, the sides of which shall be double the given base.

3. Given a straight line:—It is required to produce it so as to make it four times as long as the given straight line.

4. Given a straight line:—It is required to divide it into four equal parts.

5. Given a right angle:—It is required to divide it into four equal angles.

6. Given a straight line, and below it, to the right of it, a point A:—It is required from A to draw a straight line equal to B C.

7. Do the same, when the given point is below the given straight line, to the left of it.

8. B A C is a given isosceles triangle of which B C is the base:—Bisect the vertical angle by a straight line which intersects the base in the point D.

9. The figure of exercise 8 being drawn:—Prove that BD is equal to DC; also that the straight line A D is at right angles to BC.

10. ABC is a given isosceles triangle of which BC is the base. Bisect AB in the point D, also bisect A C in the point E. Join D C and EB:—It is required to prove that DC and EB are equal.

11. ABC is a given equilateral triangle. Bisect AB in D, and AC in E. Let the perpendiculars to AB and AC drawn from D and E meet in the point F; join FA, FB, FC:—It is required to prove that the straight lines FA, FB, and FC shall be all equal.

12. ABC is a triangle of which A is the vertex. It is given that the perpendicular drawn from A, on the base, bisects the base:—It is required to prove that the triangle ABC must be isosceles.

13. BC is a given straight line:—It is required on BC as base to construct an isosceles triangle, of which the perpendicular height shall be equal to the base.

14. Draw the figure of Euclid's fifth proposition; and where the straight lines BG and FC intersect place the letter H:—It is required to prove that BH is equal to HC. Also that FH is equal to GH.

15. In the figure of Ex. 14 join AH:—Then prove that the angle BAH is equal to the angle CAH.

16. Having proved that the angles BAH and CAH are equal (Ex. 15):—Hence prove that AH bisects BC, and is at right angles to it.

17. ABC is an isosceles triangle standing on the base BC. Bisect the angles ABC and ACB by straight lines meeting in D:—It is required to prove that BDC is an isosceles triangle, of which BC is the base.

18. ABCD is a quadrilateral figure, in which it is given that the side AB is equal to the side AD. It is also given that the straight line AC bisects the angle BAD:—It is required to prove that BC is equal to DC; and that the angle BCD is bisected by the straight line AC.

19. ACB and ADB are two triangles on the same base AB and on the same side of it. It is given that AC is equal to BD, and that AD is equal to BC:—It is required to prove that the angle DBC is equal to the angle CAD.

20. If, in the figure to Ex. 19, AD and BC intersect in the point E:—Prove that the triangle AEB, standing on the base AB, is isosceles.

21. It is given that two isosceles triangles ABC and DBC stand on the same base, and on opposite sides of it; join AD:—It is required to prove, first, that the angle BAC is bisected by the straight line AD; and hence that AD bisects BC, and is at right angles to it.

22. ABCD is a rhombus (*i.e.* a four-sided figure having all its sides equal); join AC:—It is required to prove that the angle BAD is bisected by AC. Prove also that the opposite angles BAD and BCD are equal.

23. ABC and DEF are two triangles, in which it is given (as in Prop. 8, *quod vide*), that the two sides BA, AC, are equal to the two sides ED, DF, each to each, and the base BC to the base EF. If B be placed on E, and BC be laid along EF, will C come on F? Why?—If now the triangle ABC be made to lie on the side of the common

base remote from D, and the vertices D and A be joined:—It is required to prove that the angle E A F (*i.e.* B A C) is equal to E D F.

[*Obs.* The foregoing is the proof given of the eighth prop., when Euclid's seventh proposition is omitted. To be done fully, three cases of the foregoing exercise should be considered. First, where both the angles at the base are acute; secondly, where one of them is obtuse; and lastly, where one of them is a right angle.]

24. A B C D is a given square; on one of its sides B C, as base stands an isosceles triangle E B C; join A E and D E. It is required to prove, first, that the angle A B E is equal to the angle D C E; secondly, that A E D is an isosceles triangle standing on the base A D.

25. The points O and P are the centres of two circles which intersect each other in the points A and B; join O P and A B:—It is required to prove that O P bisects A B, and is at right angles to it.

26. Let A B C D be a rectangle (that is, a figure having all its angles right angles, and its opposite sides equal); join B D:—It is required to prove that the two acute angles of a right-angled triangle are together equal to a right angle.

27. A B C is an equilateral triangle. In the sides A B, B C, C A the points D, E, F, are taken, so that A D, B E, C F are all equal. Join D E, E F, F D:—It is required to prove that D E F is an equilateral triangle.

28. From C any point in a straight line A B, C D is drawn at right angles to A B. With centre A and radius A B describe a circle intersecting C D in the point D. Join A D. From the straight line A D cut off A E equal to A C. Join E B:—It is required to prove that A E B is a right angle.

29. C B is a straight line, of unlimited length, and A is a given point without it:—It is required to find in C B a point equally distant from A and B.

30. A B is a given straight line of unlimited length, and C, D two given points:—It is required to find in the straight line A B a point equally distant from C and D. Show under what circumstances, in the 29th and 30th exercises, the solution is impossible, and under what circumstances the solutions are unlimited.

31. A and B are two points on opposite sides of the straight line C D:—It is required to find a point K in C D such that if A K and B K be joined, the angles A K C and B K C shall be equal.

32. B C is the base of a triangle, D B C is one of the angles at the base, B D is equal to the sum of its two sides:—It is required to construct the triangle. [Obs. B D must be greater than B C; that it must be so, is proved in Prop. 20.]

HINTS, AND REFERENCES TO PROPOSITIONS BY WHICH THE EXERCISES CAN BE DONE.

Exercise

1. Postulates 2, 3, Definition 15.
2. Exercise 1 and Prop. 1.
3. As Ex. 1.
4. Prop. 10.
5. Prop. 9.

6, 7. Varieties of the figure in Prop. 2.

8. Prop. 9.
9. Prop. 4.
10. Props. 10 and 4.
11. Props. 10, 11 and 4, and Ax. 1.
12. Prop. 4.
13. Props. 10, 11, 3 and 4.
14. Prop. 6, Ax. 3.
15. Prop. 8.
16. Prop. 4.
17. Props. 9 and 6.
18. Prop. 4.
19. Prop. 8.
20. Prop. 6.
21. Props. 8 and 4.
22. Part I., Prop. 8; Part II., Prop. 5.
23. Prop. 5 and Ax. 2.
24. Part I., Def. 30, Prop. 5 and Ax. 2 or 3; Part II. Prop. 4.
25. Same as Ex. 21.
26. Prop. 4.
27. Ax. 3, Prop. 4, and Ax. 1.
28. Props. 11 and 4.
29. Join A B, bisect it at right angles by a straight line, meeting C B in D. D will be the point required. Prop. 4.
30. Join C D. The rest as Ex. 29.
31. From B draw B E at right angles to C D. Produce B E to F, making E F equal to B E. Join A F and produce it to meet C D in K. K is the required point. Prop. 4.
32. Join C D, bisect C D at right angles by a straight line meeting B D in A; join A C. A B C will be the triangle required. Prop. 4.

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